Similarity through Transformations

High School Geometry 9-12

This unit focuses on understanding similarity in terms of similarity transformations and proving theorems about similarity. The emphasis shifts from traditional Euclidean geometry of expressing and using similarity (i.e., in terms of corresponding angles and sides) to transformational geometry, by building on the visual and experimental applications of dilation. Students’ prior knowledge of describing similarity through a sequence of transformations (8.G.4 & 8.G.5) forms the foundation for this unit. These concepts also build on students’ experience with the prior Model Curriculum Unit on congruency transformations, and should follow immediately after. Through guided discoveries and real-world applications, using technology and measuring tools, students learn to use their observations strategically and reason critically about their conjectures. Though the unit is written with the assumption that teachers have consistent access to technology, and that they are adept at integrating technology into their lessons, alternatives to technology are provided when possible. Heavy emphasis is placed on incrementally building students’ capacity to reason and critique the reasoning of others (Standard of Mathematical Practice 3). Students’ learning culminates with an architectural application of similarity in designing and peer-reviewing proposals for a miniature golf course.

These Model Curriculum Units are designed to exemplify the expectations outlined in the MA Curriculum Frameworks for English Language Arts/Literacy and Mathematics incorporating the Common Core State Standards, as well as all other MA Curriculum Frameworks. These units include lesson plans, Curriculum Embedded Performance Assessments, and resources. In using these units, it is important to consider the variability of learners in your class and make adaptations as necessary.
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### Stage 1 Desired Results

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<th>ESSENTIAL QUESTIONS</th>
<th>Acquisition</th>
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<tr>
<td><strong>Understand similarity in terms of similarity transformations.</strong></td>
<td><strong>Students will be able to independently use their learning to:</strong></td>
<td><strong>Understanding</strong></td>
<td><strong>Essential Questions</strong></td>
<td><strong>Students will be skilled at...</strong></td>
</tr>
<tr>
<td><strong>G-SRT.1</strong> Verify experimentally the properties of dilations given by a center and a scale factor:</td>
<td>Express appropriate mathematical reasoning by constructing viable arguments, critiquing the reasoning of others, attending to precision when making mathematical statements, and looking for and making use of structure.</td>
<td><strong>Students will understand that...</strong></td>
<td><strong>Q</strong></td>
<td><strong>Students will be skilled at...</strong></td>
</tr>
<tr>
<td>a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.</td>
<td><strong>U</strong></td>
<td><strong>Q1</strong> How can similarity be modeled in real life situations?</td>
<td><strong>K</strong></td>
<td><strong>S1</strong> Using technology to verify properties of similarity transformations</td>
</tr>
<tr>
<td>b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.</td>
<td><strong>U2</strong> Similarity can be proven through geometric transformations.</td>
<td><strong>Q2</strong> Why is the relationship between congruence and similarity significant?</td>
<td><strong>K1</strong> Targeted Academic Language:</td>
<td><strong>S2</strong> Applying theorems and properties of similarity and congruence to problem-solving</td>
</tr>
<tr>
<td><strong>G-SRT.2</strong> Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.</td>
<td><strong>U3</strong> The distance from the center of dilation to each corresponding vertex of similar figures retains the scale factor between the figures.</td>
<td><strong>Q3</strong> How do geometric transformations uncover the relationships in similar figures?</td>
<td><strong>K2</strong> Targeted Academic Language:</td>
<td><strong>S3</strong> Constructing logical arguments and forming conjectures based on observations</td>
</tr>
<tr>
<td><strong>G-SRT.3</strong> Use the properties of similarity transformations to establish the Angle-Angle (AA) criterion for two triangles to be similar.</td>
<td><strong>U4</strong> Citing evidence makes a conclusion stronger.</td>
<td><strong>Q4</strong> What is the value of logical reasoning in constructing an argument?</td>
<td><strong>K3</strong> Targeted Academic Language:</td>
<td><strong>S4</strong> Verifying experimental results of transforming figures</td>
</tr>
<tr>
<td><strong>Prove theorems involving similarity.</strong></td>
<td></td>
<td></td>
<td><strong>K4</strong> Targeted Academic Language:</td>
<td><strong>S5</strong> Applying scale factors to reason about similarity</td>
</tr>
<tr>
<td><strong>G-SRT.5</strong> Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.</td>
<td></td>
<td></td>
<td><strong>K5</strong> Targeted Academic Language:</td>
<td><strong>S6</strong> Determining whether figures are similar, using transformational geometry</td>
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### Standards for Mathematical Practice

- **SMP2** Reason abstractly and quantitatively.
- **SMP3** Construct viable arguments and critique the reasoning of others.
- **SMP5** Use appropriate tools strategically.
- **SMP 6** Attend to precision
### Connections to Literacy Standards

**R.8** Delineate and evaluate the argument and specific claims in a text, including the validity of the reasoning as well as the relevance and sufficiency of the evidence.

**W.2** Produce clear and coherent writing in which the development, organization, and style are appropriate to task, purpose, and audience.

### K2, K3, K4, S7, S8, S9

- **K2** Effects of a figure’s center of dilation on the transformation of the figure
- **K3** Relationships between similarity and congruence in triangles and geometric figures
- **K4** Geometric theorems for similarity
- **S7** Explaining the definition of similarity in terms of transformations (i.e., dilations)
- **S8** Proving relationships in figures based on theorems and observations of transformations
- **S9** Critiquing and providing feedback on each other’s reasoning

### Stage 2 - Evidence

<table>
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<tr>
<th>Evaluative Criteria</th>
<th>Assessment Evidence</th>
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| - Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar.  
- Use congruence and similarity criteria for triangles to solve problems and prove relationships in geometric figures. | CURRICULUM EMBEDDED PERFORMANCE ASSESSMENT (PERFORMANCE TASKS)  
**PT**  
A town’s revitalization committee is looking for entertainment options for the community. A survey of the town showed an interest in miniature golf. You and your team of golf course architects are designing a proposal for a golf course which you hope the town planners will accept.  
Your task is to design one hole of a mini-golf course. You are given the size and shape of the area, and the position of the starting tee. Your team is required to use at least two obstacles and at least 2 possible paths that result in a hole in one. Your proposal must include a clear explanation of how you used similar triangles to determine where to place the hole on your green, and how a hole in one is possible.  
Each team will assemble a different hole, and the collection of all class projects will constitute an entire mini-golf course. You may work in teams of no more than 3.  
**The Product**  
Your architectural team must provide to the town’s revitalization committee:  
- A 2D representation of the hole you designed and a possible path of the ball  
  - A scale drawing with obstacles included; drawn precisely with appropriate tools (ruler, etc.) |
A key for the scale drawing
- Select two similar triangles in your scale drawing and describe how you know they are similar, using transformations. Be specific, and use academic vocabulary to explain your reasoning.

- A 3D model of the green
  - The dimensions of the 3D model should be at least 8.5 x 11 inches.
  - The position of starting tee must be no farther away than 2 inches (on the scale model) from the edge of the green.
  - Locations of 2 obstacles.
  - Two (2) possible paths that result in a hole-in-one.

- A written proposal of your golf course hole, to be evaluated by the other architectural teams. Include:
  - A written justification for the location of the hole on the green, including how and which similar triangles were used in developing the design.
  - Your team’s mathematical justification for the location of the hole on the green, and how it provides two possible paths that result in a hole in one.

Precise, accurate, and thorough explanations, supported by visual transformation, demonstrating understanding of dilation as a lens to prove similarity in figures

Critique Sheet is used to guide student-to-student discussion and teacher’s formative assessment in expressing their reasoning and justifying their conclusions.

<table>
<thead>
<tr>
<th>OE</th>
<th>Triangle Similarity Project – Assessment of G-SRT.2 and G-SRT.3 (Lesson 4)</th>
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<tr>
<td></td>
<td>Dilation activity – NLVM Students explore on their own using virtual manipulatives, and follow up with conjecture-writing (based on activities in exploration). (Lesson 2)</td>
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<tr>
<td></td>
<td>Radiation: You are a dosimetrist who works for a hospital that is being sued by a patient who says that the radiation treatment he received for his illness caused damage to his spinal cord. Using similar triangles to model the position of the radiation beams, you need to prove that because the positions of the beams follow the path of two similar triangles, there could not have been damage done to the spinal cord from the radiation. (Lesson 5)</td>
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<tr>
<td></td>
<td>Given the Sierpinski Triangle, students show that the fractal is self-similar by proving that one of the smaller triangles is similar to the overall triangle. (Lesson 7)</td>
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<td></td>
<td>“Explain why, if one wants to create a similar figure with sides twice as long as the original, the angles aren’t doubled as well.”</td>
</tr>
<tr>
<td></td>
<td>Given a polygon, use the properties of similarity to fill it in with the same shape.</td>
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</table>
City Creation project:
You have just taken your dream job of becoming an engineer and your boss has just given you your first project. You have been asked to build a three-dimensional model of a city. You can be as creative as you want and may choose the materials to build your city (clay, paper, etc.). In this city you should include at least 10 building and at least 5 streets. Within these 10 buildings, there must be clear representations of two similar figures, two congruent figures, and two geometric transformations. You are required to use a straight edge when constructing your city streets. You must be able to describe the streets using the terms parallel, perpendicular, and intersecting. You are also required to determine the surface area of each building that you create. Along with this three-dimensional model of your city, you must also create a two-dimensional representation of your city. This representation should be drawn to scale. You must also include a 1-2 page write-up that will include:

- your process for creating both representations of the city,
- a description of which buildings represent which geometric concepts by using the ratio factor and the surface areas to prove that the buildings are similar,
- the city name,
- description of the streets (include street names),
- the advantages and disadvantages of two- and three-dimensional representation.

You will complete this project individually and have 2 weeks to complete the assignment. On the day that the projects are due, every student will present their project to the class explaining what they have created. Once everyone has presented, we will discuss the advantages and disadvantages of each type of representation. The write-up must include evidence of your understanding of dilation. You must prove your buildings, etc. are similar using postulates and theorems as well as factual evidence. Explanations must include the AA theorem and justification using transformational geometry.

Stage 3 – Learning Plan

Summary of Key Learning Events and Instruction

Prerequisites: Students have experience with congruency transformations. Students will also need to be reminded of the ratio and proportion work they did in 6th grade.
LESSON 1: G-SRT.1a and G-SRT.1b: Investigating Dilations
Develop the definition of dilation as a similarity transformation (see Glossary in the MA Frameworks) through construction (compass and straightedge or software, guided discovery). Using prior knowledge: recall the definitions of translations, reflections and rotations as rigid motions. Students explore the concept of dilation as a transformation.

LESSON 2: G-SRT.1a and G-SRT.1b: Exploring Dilation Using Virtual Manipulatives Dilation activity
Students explore dilations using virtual manipulatives and follow up with conjecture-writing designed to recognize important relationships in similar figures through dilation and scale factor.

LESSON 3: G-SRT.2 and G-SRT.3: Finding the Center of Dilation in Similar Figures
Students describe the properties of dilation and determine the characteristics of similar triangles; students discover the AA Theorem.

LESSON 4: G-SRT.2 and G-SRT.3: Assessment: Proving Triangle Similarity Project
Students apply their learning about similar triangles through transformational geometry, to prove similarity relationships in triangles.

LESSON 5: G-SRT.5: Applying Similarity of Triangles
Students explore the application of similar triangles in the medical field.

LESSON 6: G-SRT.2 and G-SRT.5: Justifying Similarity Using Special Triangle Patterns
The application of similarity through transformational geometry is applied to two different contexts, Sierpinski’s Triangle and a circle/triangle design.

LESSONS 7 – 9: G-SRT.5: Using Similarity to Solve Problems
Students apply their understanding of similarity through geometric transformations to investigate a variety of real-world problems, some of which may be familiar and traditionally solved by Euclidean Geometry. Students experience the shift to transformational geometry through reasoning about similarity and critiquing each other’s reasoning. They use analyses of given solutions to problems to produce their own alternative solutions. Problems include finding the height of a tree using its shadow, a “bank shot” pool table problem, and a “breakout” video game problem.
Lesson #1 – Investigating Dilations
Time (minutes): 60s
Overview of the Lesson
Students develop the definition of dilation as a similarity transformation (see Glossary in the MA Frameworks) through construction (compass and straightedge or software using guided discovery). Students use prior knowledge to recall the definitions of translations, reflection and rotation as rigid motions and explore the concept of dilation as a transformation. As you plan, consider the variability of learners in your class and make adaptations as necessary.

Standard(s)/Unit Goal(s) to be addressed in this lesson:
G-SRT.1 Verify experimentally the properties of dilations given by a center and a scale factor:
   - G-SRT.1.a A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
   - G-SRT.1.b The dilation of a line segment is longer or shorter in the ratio given by the scale factor
SMP2 Reason abstractly and quantitatively
SMP3 Construct a viable argument and critique the reasoning of others
SMP5 Use appropriate tools strategically
SMP6 Attend to precision

Essential Question(s) addressed in this lesson:
Why is the relationship between similarity and congruence significant?
How do geometric transformations uncover the relationships in similar figures?
What is the value of logical reasoning in constructing an argument?

Objectives
- Perform a dilation using appropriate technology.
- Determine how the location of the center of dilation affects the transformation of a figure.
- Determine the location of the center of dilation on the coordinate plane.
- Show that in similar figures, the distance from the center of dilation to each corresponding vertex is preserved, and shares a common scale factor.
- Articulate observations and reasoning verbally, with diagrams, and in writing, and develop criteria for critiquing each other’s reasoning.

Language Objectives

Targeted Academic Language
What students should know and be able to do before starting this lesson

Students used ideas about length and angles, how polygons behave under translations, rotations, reflections, and dilations, and ideas about congruence and similarity, in middle school to describe and analyze two-dimensional figures and to solve problems. Students will recall ratios and proportions from their work in previous grades. Through the pre-requisite Geometry Model Curriculum Unit, Congruency Transformations, students have proven congruency through rigid transformations. By now, they should have solidified formal academic vocabulary relating to congruency transformations (translation, reflection, rotation) and moved away from common language to represent the concepts (slide, flip, turn).

Anticipated Student Pre-conceptions/Misconceptions

Students may remember from middle school that dilation represents a “shrink” or “enlargement,” but they may not be aware of the precise definition: a transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor. Students are probably not yet aware of the relationship to the center of dilation and the object that is being dilated.

Instructional Materials/Resources/Tools

- Geometer’s Sketchpad or free online geometric tool (e.g., GeoGebra [www.geogebra.com] or Draw Island [www.drawisland.com])
- Handheld measuring tools (e.g., ruler, compass, protractor)
- Markers and Paper (see below) – cardstock, patty paper, transparency paper, tracing paper, and/or graph paper
- Index cards

Instructional Tips/Strategies/Suggestions for Teacher

This lesson builds a bridge between this unit on similarity (non-rigid) transformations and the prior Model Curriculum Unit on congruency (rigid) transformations. Students should recall that rigid transformations map a pre-image onto an image. In this lesson, they will use a familiar approach, but begin to notice that mapping the pre-image does not necessarily produce a congruent polygon. In this case, a different transformation has to take place… dilation.

The goal of the discussion is to get students thinking and talking about properties of similarity without putting a name to it yet. The questions foster Mathematical Practice standards SMP2 Reason abstractly and quantitatively and SMP3 Construct viable arguments and critique the reasoning of others. Students need to be explicit in their explanations. Encourage them to use evidence to support their conclusions/conjectures (SMP3), and to avoid making assumptions based solely on the diagram. Diagrams aren't always drawn to scale.

Introduction to Similarity: Familiarity with the software is a necessary prerequisite to this exploration, so that students can focus on the mathematics rather than learning the tool. Using Sketchpad or other graphing software tool supports students’ mastery of SMP5: Use appropriate tools strategically.

If needed, the first part of this exploration could be done as a whole class, in which the teacher models how to transform the triangles.
As an alternative, especially for tactile learners and/or if software is not available, this exploration can be done using tracing paper, patty paper, transparency paper, or cardstock with measuring tools (i.e., ruler, protractor).

This exploration is designed to guide students’ observations through a series of transformations. This sets up the focus for the entire unit, as opposed to similarity through angle measures and corresponding side lengths (Euclidean geometry).

You could scaffold students’ progress with prompts and sentence stems, which provides opportunities for students to put words to their observations. For example “Reflect triangle XYZ across line XZ” or “Rotating triangle XYZ about point Y results in [position of new triangle].”

You should hear some students mention similarity in their discussions of the exploration. Listen for and encourage phrases that indicate transformations (“flip”, “turn”, “slide”) as they are talking to each other, but emphasize the use of academic vocabulary (reflection, rotation, translation).

Students’ reasoning will be more qualitative and abstract at this point. Encourage them to measure angles and lengths to increase precision (SMP.6-Attend to precision).

It is important that students express their reasoning both verbally (to each other) and in writing. The Sharing and Feedback activity pushes students’ ability to provide evidence to support their conclusions and begin to learn to critique each other’s reasoning (SMP.3 Construct viable arguments and critique the reasoning of others). This higher-order thinking skill is difficult and will be revisited throughout the unit; it is introduced informally here to simply see what students can do at this introductory phase.

Critiquing the reasoning of others (SMP.3) comes with practice and by building specific criteria. Make sure that students feel comfortable with the clarity in their own articulation of reasoning before they pass it onto the next team for critique. Help ELL students access this work by providing sentence stems. For introverted students, and more intuitive students who grasp the ideas quickly but resist explaining their thinking, have them map their thinking using diagrams, and also by first giving the work a rating that represents their gut reaction (e.g., on a scale of 1-5). They should then break down the reasons for their thinking.

Guided Discovery: In this activity, discoveries are more dynamic and illustrative through the use of technology, but if not available, the work can also be done with graph paper and measuring tools (see Resources).

Students may need some visual guidance with the images they are producing, to make sure they are on the right track. Results will vary depending on which ratio each team chooses to use, but as you circulate through groups, check to make sure that students’ constructions are in line with the instructions. For those who need help, show students ahead of time, examples of possible results of transformations.

In Experiment 1: Step 3, it will appear as if nothing happened. This is a good time for discussion around the length of a line (as infinite) and how the dilation of a line maps the pre-image onto itself, in contrast to the dilation of a figure.

Students will need the time and flexibility to explore, make their own observations and conclusions, come to their own realizations and conjectures, and most importantly, to test out their ideas once they have articulated them (continue to emphasize both verbally and in writing).
Use the Additional Explorations exercises as further group work, extensions, homework, challenges for advanced students, scaffold them for struggling students, and/or as formative post-assessments. Try not to overly scaffold the work, but allow students to try possibilities and test out their ideas with different polygons or lines to reach their own conclusions and justify their reasoning. Concepts will be solidified through multiple modalities and seeing these ideas from different perspectives, and a-ha moments will emerge from seeing these ideas in a variety of ways (with technology, paper/measuring tools, and through discussion (math talk)). (SMP.5- Use appropriate tools strategically)

In the Formalizing Vocabulary conclusion, have students use the concept mapping (Lesson 1 Handout 2) activity to make connections between related terms and concepts. This activity can also be done physically by using index cards or paper cutouts to represent the open shapes. Support ELL and/or struggling students by giving them a list of terms from which to choose, checking for familiarity with the English meaning of the words in comparison to math meanings, and having them draw diagrams for each word based on the observations they made during the Guided Discovery.

Assessment
- Informal pre-assessment: Make sure students are proficient with using Geometer's Sketchpad or another geometric software tool to construct points, lines, segments and transform figures.
- Post-Assessment: See Extended Practice (below)
- Students will write conjectures based on their discovery at the conclusion of the in-class activity and then do the homework performing a dilation of a figure to reconfirm the meaning of dilation on the coordinate plane.

Lesson Details (including but not limited to:)

Lesson Opening

Introduction to Similarity: (20 min)

This lesson builds a bridge between congruence and similarity, through transformations. Facilitate a discussion using the following questions:

Do you remember what we learned about congruency transformations? What do you think is the difference between a congruence, or rigid, transformation and a similarity transformation?

You may remember from middle school that a dilation is a shrink or an enlargement, but here we will look at dilations more closely. Today we are going to use technology to explore figures that are similar, formed by dilations.

Students use technology to conduct this exploration, make observations, and form conclusions; they may work in pairs. Note: Instructions below are given in terms of the functionalities in Geometer's Sketchpad software; adjust instructions if you are using GeoGebra or another geometric tool.
Given triangle BAC [any right triangle with altitude to the hypotenuse drawn in], have students construct the area of each of the three triangles contained in the diagram. [Colors help students see the three different triangles.] Students then use the “Transformation” menu to pull the triangles apart from the original picture, while keeping track of the rigid transformations used. Precision is necessary for angle measures and lengths of translations (SMP.6-Attend to precision). Students should align the triangles so that they share a common vertex with appropriate corresponding parts.

Pose the following questions and instructions as students work on the exploration, to guide their thinking:

1. Describe the rigid transformations (or set of transformations) that mapped one figure onto another in your exploration.

Sample answer:

- Let triangle BAC = ”Triangle 1”, triangle BDA = ”Triangle 2”, and triangle ADC = ”Triangle 3”.
- Reflect Triangle 2 across BA.
- Rotate Triangle 2 about point B, ___degrees counterclockwise. [measure with protractor or use drawing tool to determine the degrees]
- This results in Triangle 2 overlapping Triangle 1 with a common vertex at point B.
- Reflect Triangle 3 across side AC.
- Rotate Triangle 3 about point A, ___degrees clockwise.
- Translate this triangle up until the uppermost angles coincide. [measure with a ruler to determine the length of translation]
- This results in Triangle 3 overlapping Triangles 1 and 2 with a common vertex at point B.

2. In the Congruency Transformations unit we used rigid transformations of triangles that resulted in the exact overlay of one triangle to the other. What do you notice about the sides opposite the common vertex here? Does this surprise you? Describe your observations and explain your thinking.

Sample answer:

The three triangles overlapped, but two sets of corresponding sides coincide, while the third corresponding sides appear to be parallel (notice the use of precise language). This is not a surprise, since the triangles are different sizes.
3. Make a conjecture that relates the triangles to each other.

*Sample answer:*

*The altitude drawn to the hypotenuse of a right triangle divides the triangle into two smaller right triangles that are the same shape as the original.*

**Sharing and Feedback (5 min)**

Have students write their conjecture on one side of an index card and the justification on the other side. They should pair up with a different partner than the person with whom they conducted their explorations. New partners give each other their cards, explain their conjectures to each other, and share feedback on their conjectures. Students may comment on accuracy, explanations, and whether sufficient evidence supports their conclusions.

**During the Lesson**

**Guided Discovery (20 min)**

Use *Lesson 1 Handout 1: Investigating Dilations through Construction.*

The purpose of this exploration is for students to consider the effects of:

- Dilating a line by a scale factor with a center of dilation on the line.
- Dilating a line by a scale factor with a center not on the line.

Students work in pairs throughout this lesson, regularly alternating their roles between manipulating the software and observing the results. Both partners should record their own results in their own words. Students will make conjectures based on their observations concerning the location of the center and the result of the dilation. This discovery is divided into Experiments. The teacher may choose to lead a whole class discussion on the findings after each or a combination of Experiments, or after the entire discovery. During a class discussion, have students project (or draw on the board) their results for reference.

Refer to *Lesson 1 Handout 1: Teacher’s Reference* for images of possible results that might arise from students’ explorations during this discovery. Instructions are written for functionalities in *Geometer’s Sketchpad*; adjust accordingly for alternative software.

In *Experiment 2*, students construct a polygon, place a point outside the polygon, and mark the point as the center of dilation. After selecting the polygon, they will transform it by any scale factor they choose. Students construct a ray from the center of dilation to a vertex of the pre-image, and then repeat for each vertex of the pre-image. By selecting a side of the pre-image to drag the figure, students will observe the meaning of dilation dynamically. **The goal is for students to begin to recognize dilation as a transformation that moves each point along a ray through the point emanating from a fixed center, and multiplies the distance from the center by a common scale factor.**

Be sure to call students’ attention to the rays and guide observations about the differences between dilating a line vs. a polygon. This definition of dilation is relatively new for students, and relates directly to the purpose of this unit – understanding the significance of similarity in terms of
transformations (i.e., dilation). Give students ample time to explore these concepts dynamically with the software, to make observations and record what they notice (with diagrams and both verbal and written explanations), and to test out their conclusions.

Questions to Guide Discussion:

- How does the distance between the center of dilation and the first polygon compare to the distance between the center of dilation and the second polygon?
- How can the two polygons coincide? (Encourage students to experiment by dragging parts of the image, pre-image, or center of dilation.)
- What transformation has occurred, and what does that tell us about the relationship between the two polygons? (Listen for ideas involving dilation as a non-rigid transformation that preserves angle measure.)

Lesson Closing

Today we learned that a dilation is “a transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor. Tomorrow we will continue our exploration of dilations in more depth. This will help us gain a deeper understanding of similarity.

Formalizing Vocabulary

At this point, it is important to formalize the definition of similarity in terms of similarity transformations. This can be done in groups of 2 or 3, and then as a whole class. Begin with a concept-mapping activity on academic vocabulary (See Lesson 1 Handout 2: Concept Mapping with Key Words). Have teams “map” their ideas (see Teacher’s Note at right). Not all shapes on the handout need to be used, and students may add shapes if needed. This activity is designed to help students make sense of the barrage of key words related to this topic, by brainstorming and sorting intuitively and through observation before writing. Conclude by having students define the terms in sentences (e.g., in a math journal) and share results as a whole class.

Additional Explorations

Discussion from the earlier Discovery activity may continue. In addition, below are further explorations to help students deepen their understanding of the notion of similarity through transformations.

- Have students repeat the procedures of the Discovery activity with a polygon of their choice, and begin to identify properties of similarity by measuring corresponding angles and sides.
- Have each member of a pair write the steps they used to make a dilation using technology; then have partners switch and see if they can follow each other’s instructions to duplicate the dilation.
- Ask students to explore the effects of transforming a parabola, by changing the symbolic expression (equation) and noticing the change in the graph, and by changing the graph and noticing the change in the equation. The Quadratic Transformer online tool performs these transformations dynamically, or a graphing calculator could be used. Ask students about the differences between dilating a line vs. parabola, what they notice
about the center of dilation, how this exploration relates to their earlier investigation, and what conjectures they can make related to their observations.

**Extended Learning/Practice (homework)**

Have each student create a polygon on the coordinate plane and multiply each coordinate by a given factor (including 2, -1, and \( \frac{1}{2} \)). Students then draw lines through the vertices of the pre-image and images to determine the location of the center of dilation.

Ask students to describe how a dilation is formed and state properties they have observed. Ask students to justify their reasoning using examples from the Discovery activity, and to explain the difference between similarity and congruence using vocabulary and concepts related to transformations.

*Sample answer: The center of dilation is the origin. A dilation is formed when rays drawn from the center of dilation intersect the vertices of the pre-image and continue for the distance of a fixed ratio to the image. A scale factor of two will increase the distance by a factor of two. A scale factor of \( \frac{1}{2} \) will shrink the distance by half. A negative scale factor will also rotate the image 180 degrees about the origin.*
Lesson 1 Handout 1: Investigating Dilations through Construction  
A Guided Discovery (using Geometer’s Sketchpad or other geometric tool)

Experiment 1: A Point and a Line

Step 1
Draw a line and a point not on the line. Using the Transform menu, mark the point as the center of dilation. Select the line and then, from the Transform menu, select Dilate by Fixed Ratio. Enter a ratio of your choice.

SAMPLE:

![Diagram of a line with a point not on the line, marked as the center of dilation, and the line selected for dilation.]

How do you think your picture would change if you changed your ratio?

Step 2
Select line AB and point C. Use the Measure menu to calculate the distance between them. Then select line A’B’ and point C, and measure the distance between them. Find the ratio of these distances.

Select both lines, one at a time, and drag the lines to observe their relationship to each other and to point C. What do you notice?

Step 3
With the Line tool selected, select point C and construct a line that contains C.
SAMPLE:

Select point C and use the Transform menu to mark C as the center. Select the line that contains C, and then, from the Transform menu, select Dilate by Fixed Ratio. What happened? Draw and explain your observations.

Based on your observations of the dilations you performed above, write a conjecture(s) to describe the relationship(s) between the point and the line(s).

Experiment 1: A Point and a Line Segment

Draw a line segment and a point that is not on the line segment.

SAMPLE:

Using the Transform menu, mark the point as the center. Select the line segment, and select Dilate by Fixed Ratio from the Transform menu. Draw a picture of what you observe:

Using the Measure menu, find the distances from C to each segment in your image. To do this, select C, and then one segment, and measure its distance. Repeat with the other line segment(s). Next select only the original line AB, and use the Measure menu to find its length. Then select only the image of AB and find its length. Record your findings.
Based on your observations, write a conjecture to describe the relationships between the point and the line segments. Describe your reasoning and support your conjecture with evidence.

**Experiment 2: Polygons**

Construct two line segments of different lengths. Select them one at a time, and use the Transform menu to mark each segment by a Fixed Ratio.

**SAMPLE:**

![Diagram of two line segments marked by a Fixed Ratio]

Construct a polygon of any size and a point not on the polygon (label the point A).

**SAMPLE:**

![Diagram of a polygon with point A marked]

Mark point A as the center, using the Transform menu. Select the polygon and, from the Transform menu, Dilate by the ratio you marked. Draw a picture of what you observe.
**Experiment 2: Dragging**

Try dragging the Center of Dilation A, dragging the endpoints of the two segments, and dragging a vertex of the polygon in a variety of ways. Can you make the two polygons coincide? How?

Drag the polygons so the figures are different sizes. Drag the center of dilation (A) onto each vertex, and compare the measures of corresponding angles. Select a side from the smaller polygon, and then select a corresponding side of the larger polygon. Then select, from the Measure menu, Ratio. Repeat to confirm this ratio with other pairs of sides. What do you notice?

How does the relationship between these two polygons compare to the relationship between any two congruent polygons? Use your conjectures and evidence from your observations to justify your thinking.

**Experiment 2: More with Polygons**

Drag the vertices of the polygons so that they are different sizes and do not overlap each other. Select point A and one vertex of the smaller polygon. Then, in the Construct menu, select Ray. Repeat with each point of the smaller polygon. Then drag to observe behavior of the polygon and ray. Measure the distance from A to any vertex of the smaller polygon. Also measure distance from A to a corresponding vertex of the larger polygon. Repeat with the other vertices.

**SAMPLE:**

Compare the ratios of these distances (A to small polygon compared to A to larger polygon). What do you notice? What transformation has occurred?

Based on your observations in Experiment 2, make a conjecture about the properties of transformational similarity. Describe your reasoning and support your conjecture with evidence.
Lesson 1 Handout 1: Teacher’s Reference

A Guided Discovery (using Geometer’s Sketchpad or GeoGebra)
The following diagrams represent possible results that students may experience during their discoveries.

Experiment 1: A Point and a Line

Step 1: SAMPLE

Student work should reflect something like the drawing above.

Step 2: SAMPLE

The ratio of the distance of line AB to point C and the distance of line A'B' to point C is equal to the ratio the student originally chose.
The distance from AB to C is in proportion to the distance from AB to A'B'.

Step 3: SAMPLE

It should appear as though nothing happened. The dilation of a line when the center of dilation is on the line results in the same line. The dilation maps the line onto itself.

Conjecture: A dilation takes a line not passing through the center of dilation to a parallel line, and leaves a line passing through the center unchanged.
Experiment 1: A Point and a Line Segment
SAMPLE

The dilation will result in a parallel segment that is larger (or smaller) by the factor the student selected.

Conjecture: The dilation of a line segment is longer or shorter in the ratio given by the scale factor. The distance of each segment from the center of the dilation is also in the ratio given by the scale factor.

Experiment 2: Polygons
SAMPLE

Students should end up with a polygon that is larger (or smaller) than the original, in the ratio of the scale factor they chose.
SAMPLE Experiment 2: Dragging

SAMPLE:
Students should notice that dragging the center of dilation moves the image in the ratio of the scale factor, and that moving a vertex on the polygon moves the corresponding vertex on the other polygon. The polygons will coincide when the center of dilation is in the center of the polygon.

Students should notice that the corresponding angles are congruent, and NOT in the same ratio as the side lengths and distance to the center of dilation. However, the side lengths ARE in the same ratio of the scale factor.

The two polygons have side lengths that are in proportion, and angle measures that are congruent. Dilation preserved the shape of the polygon, but not the size.

Experiment 2: More with Polygons
The purpose is to confirm the meaning of similarity in terms of transformations.

SAMPLE:

The ratios of the distance between point A and the vertices of each polygon is in proportion to the scale factor. A dilation has occurred.

Conjecture: Two polygons are similar if one is a dilation of the other from a fixed point.
Lesson 1 Handout 2: Concept Mapping with Key Words

Brainstorm a list of vocabulary words you encountered in *Investigating Dilations through Construction*. Sort your words into categories. Place the words related to **Similarity** on the top row (in boxes). Place the words related to **Transformations** in the bottom row (in ovals). Place any words related to the overall concept of **Similarity Transformations** in the middle (in brackets).
Lesson 2 – Exploring Dilation Using Virtual Manipulatives

Time (minutes): 60

Overview of the Lesson

Students explore virtual manipulatives, and then follow up with conjecture-writing designed to recognize important relationships in similar figures through dilation and scale factor. As you plan, consider the variability of learners in your class and make adaptations as necessary.

Standard(s)/Unit Goal(s) to be addressed in this lesson:

G-SRT.1 Verify experimentally the properties of dilations given by a center and a scale factor:
   - G-SRT.1a A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
   - G-SRT.1b The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

SMP3 Construct viable arguments and critique the reasoning of others.

SMP5 Use appropriate tools strategically.

Essential Question(s) addressed in this lesson:

Why is the relationship between similarity and congruence significant?
What is the value of logical reasoning in constructing an argument?

Objectives

- Explore the affect on the dilated image of moving the center of dilation toward or away from the original image, changing the orientation of the original figure, and changing the scale factor of the dilation.
- Recognize that dilations are not isometric.
- Show that dilated images are similar, having congruent corresponding angles and side lengths that are proportional to the original corresponding sides, by the same scale factor.

Language Objectives

Targeted Academic Language

Dilation, isometric, center of dilation, scale factor, similarity, proportion, polygon, vertices, image
What students should know and be able to do before starting this lesson

From their exploration in Lesson 1, students should know that movement of the center of dilation causes a proportional change in the distances from the center of dilation to corresponding vertices of a polygon and its image. Students should be skilled in making a dilation using technology or drawing tools, and locating the center of dilation on the coordinate plane.

Anticipated Student Pre-conceptions/Misconceptions

Students should remember the isometric transformations: translations, reflections and rotations, for which figures maintain the same distances, angles and shapes. Some students may have difficulty verbally expressing their findings and may need to use diagrams or pictures to express their conjectures. The goals are for students to articulate their thinking based on observations both verbally and in writing, to justify their thinking with evidence, and, perhaps most difficult but most important, push each other’s thinking and critique their reasoning (SMP3- Construct an argument and critique the reasoning of others).

Instructional Materials/Resources/Tools

- Computer lab, laptops, or iPads
- National Library of Virtual Manipulatives (www.nvlm.org)
- Graph paper, colored paper cutouts (optional)

Instructional Tips/Strategies/Suggestions for Teacher

Verbalizing Thinking: use this introduction as an opportunity to solidify key points from Lesson 1 and be aware of any potential misconceptions that you may need to revisit later

Resource for students needing reinforcement of concepts: http://www.youtube.com/watch?v=El7zOrCDzBs

The Online Investigation is an opportunity to continue to delve into this topic, building understanding by exploring ideas from multiple perspectives and representations. Continue to emphasize that students share their reasoning through words, pictures, and in writing. For tactile learners and/or if technology is not available, colored paper cutouts and graph paper can also be used.

Option: Assign the Conclusion questions from the Handout as homework and revisit them the next day.
Use the **Ticket to Leave** as a formative post-assessment, to gauge the conclusions that students reached. Revisit in Lesson 3, to make sure they are clearly providing their reasoning.

*About the Math...*

Students should begin to reach the following understandings about dilation by the conclusion of this lesson:

- Every point on the plane is moved to or from the center of dilation by some fixed scale factor.
- Unless the scale factor is 1, dilations are not isometric because the size is changing.
- If the scale factor is greater than 1, the image size will be larger.
- If the scale factor is less than 1, the image size will be smaller.
- If the scale factor is less than 0, in addition to the answers above, the image will be rotated 180 degrees about the origin.

**Assessment**

- **Pre-Assessment:** Use questions from Lesson 1 homework and explorations as a pre-assessment, to review and revisit their understanding of the meaning of similarity in terms of transformations and constructions using dilation.
- **Formative Assessment:** Use questions from the Conclusions section of the Investigation (Lesson 2 Handout) to check for understanding and the ability to articulate thinking and connect observations to generalizations.
- **Post-Assessment:** Ticket to Leave

**Lesson Details (including but not limited to:)**

**Lesson Opening**

*Yesterday we explored a variety of figures. We noticed that in contrast to a congruency transformation, which preserves both the size and shape, a dilation, which is a similarity transformation, maintains the shape but changes the size of the figure. Today we will continue our exploration of dilation through the use of virtual manipulatives, to see what happens when we change the center of dilation, and how the center of dilation affects the new image.*
Verbalizing Thinking (15-20 min)
Revisit guiding questions, vocabulary activity, and homework from Lesson 1. Have students pair up, switch their homework with their partners, and ask each other questions to prompt sharing their thinking. Give each partner 3 minutes (partner A asks, partner B answers, then switch). (SMP3- Construct viable arguments and critique the reasoning of others)

Sample questions that students could ask each other:
- What were your conclusions? Why?
- What observations led you to your conclusion?
- What kinds of experiments confirmed (or did not confirm) your conclusion(s)?
- My thinking was similar (or different) to yours, because…
- Does this idea make sense to you? What still seems puzzling?
- If you could do a few more test scale factors, what numbers would you try? What results would you expect? Why?

Following this pair-share, partners refine their thinking based on their discussion to formulate one new version that represents their shared ideas, and write their new conclusions on a flipchart. Do a Conveyer Belt activity (circulate each team’s flipchart to the team to the right until every team has viewed every other team’s work, 1 minute per flipchart). Each team writes comments when they receive a new flipchart (agree, disagree, circle/question mark at components that seem puzzling or need further discussion). Conclude by posting flipcharts around the room for whole class to analyze and discuss. (SMP3 Construct viable arguments and critique the reasoning of others)

Dilation is a transformation that is not a rigid transformation (isometric) because the image has a different size than the pre-image.

During the Lesson

Online Investigation (20-30 min)
In this Online Investigation, students use virtual manipulatives to gain greater familiarity with dilations (SMP5- Use appropriate tools strategically). Allow students a couple of minutes to play around with the site and the online tool prior to completing the handout, so that their lack of familiarity with the tool will not interfere with their exploration of the concepts. The handout is designed to guide student exploration while maximizing independent discovery. Results may vary, but students should be reaching similar conclusions about relationships being observed. Have students work in pairs; conclude with whole class discussion.

Suggested Discussion Questions:
- How does the location of the center of dilation influence or change the relationship between figures being observed (pre-image, image)?
- How would your investigation have been different if the images were congruent? Why is the distinction between congruency and similarity important here (Essential Question)?

**Lesson Closing**

*From our exploration today, we noticed the importance of the scale factor. Depending on whether the dilation scale factor is greater than or less than 1, it will affect the size of the figure differently. Tomorrow we will continue to use similarity transformations, this time to discover more specific relationships in similar figures, specifically triangles.*

**Ticket to Leave (5 min)**

Students write responses to the following questions. Collect before they leave.

**Through a dilation,**

1. If the scale factor is greater than 1, the image size will be __________ [larger]
2. If the scale factor is less than 1, the image size will be __________ [smaller]
3. Explain your reasoning for your answers to 1 and 2.
   
   [When I multiplied by a number greater than 1, the vertices increased from the origin by that scale factor. When I multiplied by a number less than one, the vertices were shrunken by that scale factor.]

What if the scale factor is less than 0? Describe what happens, and how you know. [The image will be larger or smaller depending on the absolute value of the scale factor (see above), but the whole image will be rotated 180 degrees about the origin.]

**Extended Learning/Practice (homework)**

Real-World Connection: Application of Scale Factors

Where would you see or use scale factor in everyday life? Write a paragraph, using some of the key terms we have been learning in this Similarity Transformations unit that describes the connections you see between your real-life example and the mathematical concepts. Include diagrams, written explanation of your reasoning, and support your conclusions with evidence.
Lesson 2 Handout: Investigating Dilation Using Virtual Manipulatives
This investigation will be conducted online. Go to the National Library of Virtual Manipulatives website (http://nlvm.usu.edu/). Select Geometry, and then Grades 9-12, and then Transformations – Dilations.

Take some time to explore the available options in the online tool. When you are ready, complete the following experiments and describe your observations.

1. **Move the center of dilation (black dot).**

In relation to the center of dilation, what happens to the copied image as the black dot is moved toward or away from the original image? Why do you think this occurs?

2. **Predict:** What will happen to corresponding parts of the copied image if you rotate the original image?

**Change the orientation of the original figure.** Describe your observations. Was your prediction correct? Why do you think so?

3. **Change the scale factor of the dilation (use the slider).** Try moving the slider to the following values: 0.25; 0.50; 0.75; 1.0.

At each of these values, what do you notice about the size and location of the image in relation to the original? How would you explain these changes?

4. **Click the box to the right of the word “axes.”** Select a shape (you could try the square first, since it is easy to see). Place the shape on the coordinate plane so that the bottom left-hand vertex of the shape is located at the origin. Now rotate the original shape.
What did you notice?

How are the slopes of the corresponding sides of the original figure related to the slopes of the sides of its image? Explain.

Conclusions

5. What is a scale factor?

6. Make a generalization about the relationship between the copied image and the original image. Explain your reasoning. What could this relationship help you to prove?

7. How does the location of the center of dilation of a figure affect its resulting image? Use your observations to justify your conclusion.
Lesson 3 – Finding the Center of Dilation in Similar Figures

Time (minutes): 60

Overview of the Lesson
Students describe the properties of dilation and determine the characteristics of similar triangles; students discover the AA Theorem through transformations. As you plan, consider the variability of learners in your class and make adaptations as necessary.

Standard(s)/Unit Goal(s) to be addressed in this lesson:

G-SRT.2 Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

G-SRT.3 Use the properties of similarity transformations to establish the Angle-Angle (AA) criterion for two triangles to be similar.

SMP2 Reason abstractly and quantitatively.

SMP5 Use appropriate tools strategically.

Essential Question(s) addressed in this lesson:

How can similarity be modeled in real life situations?
Why is the relationship between similarity and congruence significant?

Objectives

- Use similarity transformations to support the AA postulate.

Language Objectives

Targeted Academic Language

Theorem, postulate

What students should know and be able to do before starting this lesson

Students should have a basic understanding of similarity and a thorough understanding of dilation from previous lessons. They should also be familiar with the triangle congruence theorems, and they should recall the Angle Sum theorem for triangles from working with Geometry in middle school.
Anticipated Student Pre-conceptions/Misconceptions

This lesson is language-heavy. Students may struggle with writing explanations, more so than in prior lessons.

Instructional Materials/Resources/Tools

- Ruler and protractor
- Geometer's Sketchpad or free online geometric tool (e.g., GeoGebra www.geogebra.com or Draw Island www.drawisland.com)

Instructional Tips/Strategies/Suggestions for Teacher

Whereas earlier lessons were more exploratory, students will now begin to focus their explanations more sharply and the importance of justifying their reasoning with evidence becomes even more significant (SMP3). This lesson also introduces a real-world application for similarity.

In this lesson, students are required to explain much in writing about the transformations they are using. Before they conjecture about similarity theorems, students should gain familiarity and comfort with writing informal proofs of congruency theorems.

In the Warm-Up, it should be fairly clear that the center of dilation is in the center of the polygons. Ask students if this would be the case if the figures were not regular.

In the Discovery Investigation, students are given two similar rectangles not in the same orientation. They should discover through drawing lines between corresponding vertices that there is no center of dilation, and therefore, one figure is NOT a dilation of the other.

It would be good to review the definition with students at this point: Dilation is defined by the CCSS as “a transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor.”

The Whole Group Exploration is entirely teacher-led. Emphasis should be placed on listing the detailed rigid transformations necessary to align the triangles (in order to create a center of dilation). Note: arranging the figures so that they share a corresponding angle makes finding the center of dilation a little easier.
For students who struggle with language: have steps written out ahead of time: “rotate figure”, “reflect figure”, “translate figure”, “dilate figure”, and have the students put the strips in order.]

The end of this investigation leads students to the AA Postulate.

In the Interactive Investigation, students may discover multiple transformation steps to align the two triangles. Some sequences result in more steps than others (for example, students might reflect first, then rotate, then translate, in which case they may have to rotate again). Discuss ways they can be efficient with the transformation sequence.

Assessment

- Pre-Assessment: Warm-Up 1. Sharing students’ responses from Lesson 2, to gauge the use of vocabulary and how students are making connections between the concepts
- Formative assessment: Warm-Up 2.
- Formative assessment: Last question of the Discovery Investigation requiring students to write out the sequence of transformations.

Lesson Details (including but not limited to:)

Lesson Opening

Before we begin, let’s collect our understandings from the past few lessons. A dilation preserves the shape, but not the size, of a figure, but the change in size is determined by a specific ratio, called the scale factor, and by the location of the center of dilation. Keep this in mind, as today we investigate a specific relationship between the angles in similar triangles. You may remember from middle school, theorems about triangles. For example, if you know the measures of 2 angles in a triangle, how could you find the third? (They all add up to 180) That's an example of a theorem, the angle-sum theorem, or postulate. Today we’re going to learn a new postulate.

Warm-Up (10 min)

1. Revisit students’ writing about real-world examples of similarity/dilations from Lesson 2. Have students post their work around the room and do a Gallery Walk to read and respond to each other’s writing. Share as a class.
2. Have students use the picture of the Pentagon (below) to, first, describe/predict the relationship between the pentagon created by the innermost wall and the pentagon created by the outer wall. [Students need to find the center of dilation and use the distance from it to each corresponding side (or vertex) to establish a scale factor.]

![Pentagon Picture]

*Discuss:* What properties did you need to consider in demonstrating similarity, and how do these relate to the definition of similarity? What is the difference between showing (or proving) and explaining? Is the transformation dilation? How can you tell?

**During the Lesson**

*Discovery Investigation, 10 min*

Students are asked to find the center of dilation given two similar non-square rectangles that are not in the same orientation. They will need to construct lines through corresponding vertices. They should notice that the lines do not intersect in a single point, thereby verifying the need to rotate the shapes before performing the dilation.

Q: What rigid transformation(s) will place the two rectangles in the same orientation?
Q: Is there a correct order of transformations to align similar shapes so that you can find the center of dilation?

**Whole Group Exploration, 20 min:**
Provide students with a copy of Lesson 3: Handout 2, the Illustrative Math activity, G-SRT.3: Similar Triangles. As a whole group, guide students through the process of aligning the two figures, and then finding a center of dilation. Use the figures shown, and then write the steps in a list separately.
Q: Is this the only way to orient the figures? In what other ways can this be done?
Q: What does this investigation uncover about triangles with pairs of congruent angles?

**AA Postulate:** If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

**Interactive Investigation #2 (15 min)**
Students will engage in further discovery using Geometer’s Sketchpad or alternative software (see Resources). The connection to the AA postulate is not yet explicit. Use Lesson 3: Handout 3, Investigating Similarity through Construction. Instructions are written for Geometer’s Sketchpad; adjust accordingly for alternative programs.

Remind students about the importance of justifying their reasoning (SMP3) and record their thinking, both verbally and in writing. Encourage students to use a Math Journal.

*Questions to Guide Discussion:*
- How is transformation a valuable approach to understanding similarity? How would our understanding of similarity be limited without seeing the connections through transformations/dilation?

**Practice (Optional)**
This may be a good time for practice, review, and building the skill of using proportional relationships to apply algebraic concepts to properties of similarity. Feel free to engage students in traditional problems that involve solving for the missing side in triangles, based on a set of given measurements. Use these as practice, homework, and/or for struggling students.

**Lesson Closing**
Have a few students summarize in their own words and share with the whole class, the AA Postulate they investigated today. How is this related to our ongoing investigations of similarity through transformations? Tomorrow, we’re going to formalize your learning about similarity and transformations and you will have an opportunity to use properties of dilation to prove that 2 figures are similar.

Ticket to Leave
In the figure shown,

1. Which two triangles are similar? Justify your reasoning using a transformation sequence.
2. Set up a proportion that you can use to find the value of x.
3. Use your proportion from question 2 to find the value of x.
Lesson 3, Handout 1: Discovery Investigation

Directions:

1. Find the center of dilation given the two rectangles by constructing lines through corresponding vertices. Is it possible? Why or why not?

2. Perform rigid transformations to align the rectangles. List the transformation(s).

3. Find the center of dilation of the re-aligned rectangles.
Lesson 3, Handout 2: Similar Triangles

Adapted from Illustrative Mathematics Project

In the two triangles pictured below $m(\angle A) = m(\angle D)$ and $m(\angle B) = m(\angle E)$

Using a sequence of translations, rotations, reflections, and/or dilations, show that $\triangle ABC$ is similar to $\triangle DEF$. 
Lesson 3, Handout 3: Investigating Similarity through Construction by AA
A Guided Discovery (using Geometer’s Sketchpad or other geometric tool)

We have learned that for a pair of triangles to be similar, corresponding pairs of angles need to be congruent and corresponding pairs of sides need to be proportional. In this experiment, we will investigate a short-cut.

1. Construct a triangle and a line not on the triangle.

SAMPLE:

2. Double-click on point E to mark it as the center of rotation. Select point A, B, C (in this order), and then use the Transform menu to mark it as the angle of rotation. Select only line DE, and then use the Transformation menu to rotate.

SAMPLE:

3. Double-click point D to mark it as the center of rotation. Select point A, C, B (in this order), and then use the Transform menu to mark it as the angle of rotation. Select only line DE, and then use the Transformation menu to rotate.
4. Construct the point of intersection of these lines (label it F). Select all three lines and use the Display menu to “Hide Lines.”

SAMPLE:

5. Select the three points and use the Construct menu to construct segments and form the triangle connecting D, E, and F. Select a side from triangle ABC, and then a corresponding side from the other triangle. Use the Measure menu to find the ratio. Repeat this with the remaining sides.

SAMPLE:

\[
\frac{m_{AB}}{m_{EF}} = 0.86
\]

Conjecture what the Angle-Angle Postulate says about triangles with two pairs of angles congruent. Justify your reasoning.

Using a sequence of translations, rotations, reflections, and/or dilations, show that your two triangles are similar. (This may be easier with patty paper and pencil.)
Lesson 4 – Assessment: Proving Triangle Similarity Project
Time (minutes): 60
Overview of the Lesson
Students apply their learning about similar triangles through transformational geometry to prove similarity relationships in triangles. As you plan, consider the variability of learners in your class and make adaptations as necessary.

Standard(s)/Unit Goal(s) to be addressed in this lesson:
G-SRT.2 Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
G-SRT.3 Use the properties of similarity transformations to establish the Angle-Angle (AA) criterion for two triangles to be similar.
SMP2 Reason abstractly and quantitatively.
SMP3 Construct viable arguments and critique the reasoning of others.
SMP5 Use appropriate tools strategically.
W.2 Produce clear and coherent writing in which the development, organization, and style are appropriate to task, purpose, and audience.

Essential Question(s) addressed in this lesson:
Why is the relationship between similarity and congruence significant?
What is the value of logical reasoning in constructing an argument?

Objectives
Use the properties of dilation to prove triangle similarity.

Language Objectives

Targeted Academic Language

What students should know and be able to do before starting this lesson
Students understand dilation as a transformation, resulting in similar triangles with proportional corresponding angles and corresponding sides that are proportional by a given scale factor.
Students have mastery of all related academic vocabulary: dilation, center of dilation, scale factor, similarity, proportion, orientation, polygon, vertices and image.

**Anticipated Student Pre-conceptions/Misconceptions**

Students may forget that dilations are not isometric. Students should recall that dilated images are similar, having congruent corresponding angles and side lengths that are proportional to the original side lengths by the same scale factor.

**Instructional Materials/Resources/Tools**

- Ruler, graph paper, and protractor
- *Geometer's Sketchpad* or free online geometric tool (e.g., *GeoGebra* [www.geogebra.com](http://www.geogebra.com) or *Draw Island* [www.drawisland.com](http://www.drawisland.com))
- Resources for students needing reinforcement of concepts: [http://www.youtube.com/watch?v=El7zOrCDzBs](http://www.youtube.com/watch?v=El7zOrCDzBs) [http://www.youtube.com/watch?v=CLIudjQDebo](http://www.youtube.com/watch?v=CLIudjQDebo)

**Instructional Tips/Strategies/Suggestions for Teacher**

This assessment addresses content and literacy objectives. Understanding and using academic vocabulary can help learners make sense of a problem. Mathematically proficient students should be able to explain their reasoning to themselves and to others.

Students need to use language to work through problems, communicate their thoughts, and structure their arguments. They need options for expressing themselves in a manner that most suits their strengths. This project provides these opportunities while inviting creativity and expression/presentation skills.

Students will explain, verbally and in writing, the logical reasoning process used to complete their project (MP2). They will need to be able to defend their conjectures and listen to and write about the reasoning of others (SMP3- *Construct viable arguments and critique the reasoning of others*). By now, students should be more adept at drawing triangles and measuring side lengths and angles using geometric software tools and/or paper, pencil, and protractor (SMP5- *Use appropriate tools strategically*).

**Assessment**

Formative assessment: Warm-Up (homework review)
Proving Triangle Similarity Project

**Lesson Details (including but not limited to:)**
Lesson Opening
Assessment; see below

During the Lesson

Warm Up (5 min)
Review Homework from Lesson 3 to check for understanding of scale factor (Ask: “What is the scale factor between the two similar triangles?”), and the relationships between an original figure and its image.

Proving Triangle Similarity Project (50 min)
This formative performance assessment will be conducted in teams of 2. Use Lesson 4 Assessment: Proving Triangle Similarity Project.
Stress the importance of collaboration as a means of seeing something from a different perspective, as we continue to build the critical skill of critiquing each other’s reasoning by learning to clearly express our own (SMP3). Urge students to talk to each other about the measurements they are taking, the conclusions that can be made from their data, and how this data can be used with the AA Postulate to prove or disprove similarity (SMP2 Reason abstractly and quantitatively).

Problem 1 asks students to provide evidence of their learning about transformations to determine and verify the relationship between given similar triangles. Problem 2 asks student to think backwards, in using their understanding of transformation/dilation to create similar triangles. It is important for students to think flexibly (21st Century skills) by considering one option and then vice versa, and demonstrating evidence of their reasoning.
(SMP3- Construct viable arguments and critique the reasoning of others)

Discuss: Are the relationships between the triangles preserved if you move or change the dimensions of a triangle? How do you know?
What is the scale factor of the two triangles? Is there a dilation? Why or why not? If so, what is the center of dilation?

Ensure that students are on task, successfully working through the problems, understanding the meaning of similarity and transferring their knowledge to the assigned explanations. Reiterate the requirement that each partner is able to explain his/her own reasoning with supporting evidence. When teams have completed their own projects, they will critique another team’s reasoning (SMP3- Construct viable arguments and critique the reasoning of others). They will conclude with presentations of their own work and their critique of another team.

Lesson Closing
Summarize the lesson
Lesson 4 Assessment Handout: Proving Triangle Similarity Project

Part 1: Constructing Triangles

Complete Problems 1 and 2 with a partner, using a ruler.

You will use the pairs of triangles that your team constructs in Problems 1 and 2 for the Proving Triangle Similarity Project (see instructions on the next page).

**Problem 1** – Prove that the following triangles are similar, using your understanding of transformational geometry. Explain your reasoning in words and demonstrate using clear visuals. (If it helps, label each triangle.) How would you verify your reasoning?

**Problem 2** – Construct two triangles that are similar, using dilation. Justify your approach using transformational geometry.
Part 2: Project Instructions

Using your results from Problems 1 and 2, your goal is to **prove or disprove the similarity of the pair of triangles you constructed.** You will use the data you collected and your understanding of similarity and its associated theorems.

**Team Presentation (SMP2 Reason abstractly and quantitatively)**
- Trace your triangles from Problem 2 onto a blank sheet of paper, without any guiding lines (trace the triangles only).
- Give the sheet to another group. You will be given a sheet with two triangles as well.
- Show that the two triangles you are given are similar using similarity transformations. **Your team will present your results to one other team.** Your presentation should include **all** the following components:
  - Clearly labeled diagrams of your triangles
  - A table listing the angle measures (measure with a protractor) and side lengths of your triangles
  - The scale factor of dilation
  - Proof of similarity for your triangles:
    - Using the definition of similarity through transformations
    - Using the AA Postulate

**Criteria for Success**
Clear communication between partners will ensure your success on this assignment. As you work together, be sure to let your partner know if you do not agree with the process or the calculations being made. Aim for precision. All statements, measurements, and calculations should be accurate. As presenters, both you and your partner(s) must be able to justify and communicate your conjectures, explain the logical progression of thoughts that led to your conclusions, and respond to any questions or arguments that might arise (SMP2: Reason abstractly and quantitatively; SMP3- Construct viable arguments and critique the reasoning of others).

**Team Critique and Written Synopsis**
Your team will be required to critique one other team’s presentation, using the Critique Sheet. As observers, be sure to listen to their reasoning closely, decide whether or not it makes sense, and ask questions to clarify or invite them to elaborate on their arguments.

Provide, with all relevant evidence,
- A definition of similarity in terms of transformations (in reference to triangles), written **in your own words**, including appropriate academic language (vocabulary)
- A written description of the logical progression of your team’s solution process, your conclusions, and justifications of your team’s reasoning
- Your team’s critique and observations of the other team’s presentation that you observed
Extra Credit: Explain why, if you want to construct a similar figure with sides twice as long as the original figure, the angles are not also doubled. Include a diagram, if needed.

Critique Sheet

Use this sheet to critique another student or team's problem-solving work. Explain your rationale in the comments section as clearly as possible.

Your/Your Team's Names: __________________________________________________________

Student/Team's reasoning is clear: yes no (circle one)

Comments:

Student/Team's reasoning is logical: yes no (circle one)

Comments:

Student/Team's mathematical thinking/calculations are accurate (circle one):

consistent mostly not consistently not at all
Lesson 5 – Applying Similar Triangles

Time (minutes): 60

Overview of the Lesson

Students explore the application of similar triangles in the medical field. As you plan, consider the variability of learners in your class and make adaptations as necessary.

Standard(s)/Unit Goal(s) to be addressed in this lesson:

G-SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.
SMP2 Reason abstractly and quantitatively.
SMP3 Construct viable arguments and critique the reasoning of others.
R.8 Delineate and evaluate the argument and specific claims in a text, including the validity of the reasoning as well as the relevance and sufficiency of the evidence.
WHST.10.9 Draw evidence from informational texts to support analysis, reflection, and research.

Essential Question(s) addressed in this lesson:

How can similarity be modeled in real life situations?
What is the value of logical reasoning in constructing an argument?

Objectives

- Identify similar triangles using either AA or a sequence of similarity transformations.
- Solve real-world problems involving similar triangles.
- Apply other students' understanding of similarity and theorems governing proportional relationships to real-world situations, and interpret their conclusions in context.

Language Objectives

Targeted Academic Language
Dosimetry
What students should know and be able to do before starting this lesson
Students have been working with similarity and similarity transformations. They should be familiar with determining whether two figures (i.e., triangles) are similar through similarity transformations, and using AA to prove conjectures about similar triangles and their properties.

Anticipated Student Pre-conceptions/Misconceptions
Watch for the ways that students interpret the relationships in given problems, and how they set up proportions to determine unknown lengths, as students may set up incorrectly depending on their interpretation.

Instructional Materials/Resources/Tools
Handheld measuring tools (e.g., ruler, compass, protractor)

Instructional Tips/Strategies/Suggestions for Teacher

Students experimented with properties of similar triangles and proof of similarity in the previous lesson. In this lesson, they will apply the properties of similar figures to solve problems. Students are introduced to applications of similarity in context. This lesson provides a real-world example of the use of similarity in medicine and engages students in investigating the Essential Question, modeling similarity in real-life situations.

Warm-Up problem reviews and scaffolds the process of determining corresponding parts of the similar triangles before setting up and solving the proportion. In addition, students are asked to show similarity through transformations.

By this point, students have had multiple experiences with discovering the meaning of similarity in different representations and perspectives and articulating their thinking, justifying their reasoning, and critiquing each other’s reasoning. Take some time for students to reflect on and discuss the Essential Question about constructing arguments and critique (SMP2- Reason abstractly and quantitatively; SMP3- Construct viable arguments and critique the reasoning of others), and how these investigations may be pushing their thinking in different ways. Continue to emphasize expressing reasoning verbally and in writing.
**Dosimetry: (according to Wikipedia)** Radiation dosimetry is the measurement and calculation of the radiation dose received by matter and tissue resulting from the exposure to indirect and direct radiation.

The **Medical Application** problems (Lesson 5 Handout) push students’ ability to critique each other’s reasoning a bit further than in prior lessons, moving from expressing one’s own reasoning to articulating another’s reasoning from their perspective to formulating a new conclusion based on the value found in both partners’ reasoning.  (SMP3- *Construct viable arguments and critique the reasoning of others*) Students may need some scaffolding in describing the situation before they tackle the first problem.

In the **Shadow Problem**, watch out for incorrect measurement conversions (5’4” to 5.4 feet). Also watch for incorrect proportions. If students are finding Nancy to be shorter than Michelle, refer them back to their answers to part a.

**Assessment**
- Formative Assessment: Warm-Up problem
- Formative Assessment: Gauging students understanding and application of proportional relationships through real-world problem-solving (through discussion questions)
- Post-Assessment: Ticket to Leave

**Lesson Details (including but not limited to):**

**Lesson Opening**

*We have been building an understanding of the power of transformations to understand, analyze, and prove the relationship between 2 similar (and congruent, from prior learning) figures. We will continue to use this lens of transformation as we explore a real-world application of similarity in the medical field. Here we will see how the AA Postulate, the concept of dilation, and the related ideas of scale factor and proportional relationships come alive.*

**Warm-Up/Pre-Assessment (10 min)**
- Give students the following word problem in which they need to find the missing length in a set of triangles that are similar. Gauge students’ application of proportional relationships in interpreting the problem.
Two ladders are leaning at the same angle against a vertical wall. The 3 meter ladder reaches a point on the wall 2.5 meters from the ground. How much higher on the wall does the 5 meter ladder reach?

Sample Answer: Since the ladders represent the hypotenuse in each triangle we can find the corresponding height on the wall by setting up a proportion: \[
\frac{3}{2.5} = \frac{5}{x} \quad x = 4.17, \quad 4.17 - 2.5 = 1.67
\]
It will reach 1.67 meters higher on the wall.

Ask students to draw a diagram of the problem, and how they know that the two triangles created are similar. Ask them what rigid transformations are needed to show that the two triangles in the word problem are similar.

Sample Answer: Since the wall is vertical, both ladders create a right angle. We are told that the ladders are leaning at the same angle, therefore by AA, the triangles are similar.

Discuss results from prior Assessment (Proving Triangle Similarity Project). Ask students what they found useful about the experience, what questions they still have, and their thoughts about the Essential Question, “What is the value of logical reasoning in constructing an argument?”

During the Lesson

Applying Similarity: Dosimetry (20-30 min)
Students (individually) work on Lesson 5 Handout: Medical Application of Similar Triangles. Remind students to record and provide evidence for their reasoning. They will share their reasoning with at least two other students in Problem 5.

Discussion Questions:

- Why does the concept of similar triangles lend itself so well to this problem? [The radiation beam creates a situation that forms a dilation.]
- Which is more difficult, articulating your own reasoning or your partner’s reasoning? Why? What criteria did you use to critique each other’s work?
Shadow Problem (10 min)
Project, or provide a handout with, the following problem for students to consider:

On a sunny day, Michelle and Nancy noticed that their shadows were different lengths. Nancy measured Michelle’s shadow and found that it was 96 inches long. Michelle then measured Nancy’s shadow and found that it was 102 inches long.

a. Who do you think is taller, Nancy or Michelle? Why?

*Sample Answer: Nancy must be taller because she casts a longer shadow*

b. The sketch below (not drawn to scale) shows Nancy, Michelle and their shadows. Label the drawing appropriately.

![Sketch of shadows](image)

c. Are the triangles shown similar? How do you know? What is the scale factor?

*Sample Answer: The triangles are similar because they each have a right angle and they share another angle, so by AA they must be similar. The scale factor is \( \frac{102}{96} \) since that is the ratio of the corresponding sides.*

d. What transformation sequence will align the smaller triangle with the larger one?

*Sample Answer: A dilation with scale factor 102/96.*

e. If Michelle is 5 feet, 4 inches tall, how tall is Nancy?
Sample Answer: 5 feet, 8 inches; the proportion is \( \frac{96}{64} = \frac{102}{x} \). \( x = 68 \)

f. If Nancy is 5 feet, 4 inches tall, how tall is Michelle?

Sample Answer: 59.1 inches; the proportion is \( \frac{102}{64} = \frac{96}{x} \). \( x \approx 59.1 \)

Lesson Closing

Ticket to Leave (5 min)
Although the problems we explored today represent different real-life situations, they both applied important concepts about similar triangles. What were some key elements of the problem-solving process that resembled each other in terms of using similar triangles to find a solution?

Have students write about this question:
What other kinds of real-world applications of similarity, particularly in triangles, can you imagine? Give at least one example.

Homework
- A statue of a local hero is in a park. Victoria would like to know the height of the statue but it is too tall for her to measure. Tell how Victoria can determine the height of the statue using a tape measure, the lengths of shadows, and similar triangles. Explain your reasoning.

Sample Answer: Victoria can measure the lengths of the two shadows and her own height and set up a proportion.

- The length of the shadow cast by the statue is 17.5 feet. The length of the shadow cast by Victoria is 7 feet. Victoria is 5.5 feet tall. How tall is the statue? (Draw a diagram of the problem).

Sample Answer: 13.75 feet; the proportion is \( \frac{7}{5.5} = \frac{17.5}{x} \). \( x = 13.75 \)

- What is a series of transformations that would map one triangle in your diagram onto the other?
Sample Answer: A translation and then a dilation (depending upon students’ drawings)

- Why do you know that the triangles in the diagram are similar?

Sample Answer: Both Victoria and the statue create right angles, and the sun is at the same angle on the horizon so the triangles are similar by AA.
Lesson 5 Handout: Medical Application of Similar Triangles

In today’s professional world, there are many practical applications that rely on triangle similarity. One such application is in medicine, where similar triangles are used to calculate the position of radiation treatment for cancer patients. Dosimetrists need to correctly calculate the position of radiation beams so as to avoid their overlap on the tumor location. They need to avoid giving too high of a dosage of radiation on the skin’s surface. This is possible by adjusting the sources so that the overlap only occurs at the point of the tumor.

A particular tumor is requiring two simultaneous doses of radiation given from sources at a fixed distance. The position of radiation beams can be approximated by the figure (not to scale). The patient is having radiation treatment on his back.

The source of radiation is a fixed distance of 1000 mm for all patient treatment. This creates two similar triangles (shown in red).

1. Describe the center of dilation for the two similar triangles.

If radiologists know the depth from the skin to the tumor (in the above picture, this distance is 5 mm), they can calculate the distance that the beams should be set apart. They use similar triangles to determine the unknown horizontal distance.

2. How do they know that the two triangles are similar?
3. How far apart should the beams be in the diagram above (distance x)?

4. Another patient is having radiation therapy on his tibia (shin bone). The horizontal distance of the field is measured at 7.97 mm. The beams are set to be 16 mm apart. How deep does the field penetrate the surface of the leg (distance y)?

5. Share your solution to Problem 4 with two other classmates. Did you use the same method for solving? What was different? Did you get the same solution? If not, try to come to an agreement about the “correct” solution. Record your reasoning, your partner’s reasoning, and your agreed-upon conclusions.

<table>
<thead>
<tr>
<th>My Reasoning</th>
<th>My Partner’s Reasoning</th>
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Our Joint Conclusion(s) and Reasoning

Sample Response:

1. The center of dilation is the source of the radiation.
2. The radiation is beamed at an angle. The similar triangles share this angle and the right angle created by the center of the beam, therefore the triangles are similar.
3. 24.12 mm; to find the distance to the center of the tumor I set up a proportion: 1000/12 = 1005/d. I solved for d (12.06) and doubled that distance to find x.
4. 3.764 mm; I set up a proportion and solved: 8/7.97 = (1000 + y)/1000
Lesson 6 – Justifying Similarity Using Special Triangle Patterns

Time (minutes): 60

Overview of the Lesson
The application of similarity through transformational geometry is applied to two different contexts, Sierpinski’s Triangle and a circle/triangle design. As you plan, consider the variability of learners in your class and make adaptations as necessary.

Standard(s)/Unit Goal(s) to be addressed in this lesson:
G-SRT.2 Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
G-SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.
SMP2 Reason abstractly and quantitatively.
SMP5 Use appropriate tools strategically.
W.2 Produce clear and coherent writing in which the development, organization, and style are appropriate to task, purpose, and audience.

Essential Question(s) addressed in this lesson:
How can similarity be modeled in real life situations?
Why is the relationship between similarity and congruence significant?

Objectives
• Apply properties and similarity to fractals.
• Apply students’ understanding of similarity and theorems governing proportional relationships to complex mathematical phenomena.
• Solve problems by determining the lengths of the sides in right triangles.
• Find the measurements of shapes by decomposing complex shapes into simpler ones.
• Recognize that there may be different approaches to geometrical problems, and understand the relative strengths and weaknesses of those approaches.
Language Objectives

Targeted Academic Language
Iteration, inscribed, fractal

What students should know and be able to do before starting this lesson
Students understand that similar triangles have the same shape although different sizes, with corresponding angles equal and corresponding sides proportional. Students should be gaining a clearer understanding of similarity and related vocabulary, and developing greater comfort and skill in explaining and justifying their conjectures. They should know that the change in the length of the sides of the triangle is caused by multiplication by a scalar (fixed ratio).

Anticipated Student Pre-conceptions/Misconceptions
Students may be tempted to make conclusions based on the diagrams, without paying close attention to evidence from observable, measurable properties, characteristics, or data. In addition, during this lesson, students may make calculation errors computing ratios, or they may struggle with explaining their solution methods clearly in writing.

Instructional Materials/Resources/Tools
Paper, pencil, ruler, and protractor

Instructional Tips/Strategies/Suggestions for Teacher
This lesson uses a protractor and ruler to gain a visual and tactile understanding of the physical model, but it can (and should) also be completed using Geometer’s Sketchpad (or alternative geometric software). Each modality provides a different entry point to deepen student learning. (SMP5 - Use appropriate tools strategically)

Students should be able to see the congruence of newly formed triangles, and be able to justify the similarity between the four congruent triangles formed in the first iteration of the triangle and the original figure.

This lesson may be implemented as a group activity, with either heterogeneous or homogeneous grouping. You may want to have students complete two or more iterations and report on their findings first, and then make conjectures about the results of further iterations.
Students have been familiar with recursion since Kindergarten, but may not have realized it. This is an opportunity to bring out the depth of recursive patterns and arithmetic and geometric sequences, by making exciting connections to algebra through the use of fantastic visuals.

The Transformations Using Triangles and a Circle problem is provided to students ahead of time to allow students a preview if they need it. Some students may have difficulty getting started during the Warm Up.

Source: This problem is adapted from the Mathematics Assessment Project: Classroom Challenges lesson called “Geometry Problems: Transformations Using Triangles and a Circle.” Modifications to the original lesson have been made to adapt to the needs of the overall model unit design.

Introduction to Transformations Using Triangles and a Circle: It is not important that students come to a solution at this point, just that they have thought about the problem and have begun a plan for solving it. The questions to think about are optional and serve as supports for students who may be stuck.

Problem-Solving Collaboration: During the lesson, students work collaboratively to solve the problem. While students work in small groups, you have two tasks – to note their different approaches to the task, and to support/draw out their reasoning.

Note any errors, and think about your understanding of students’ strengths and weaknesses from the task. You can use this information to focus whole-class discussion towards the end of the lesson.

Try not to make suggestions that move students towards a particular approach to the task. Instead, ask questions that help students to clarify their thinking and express their reasoning (SMP2- Reason abstractly and quantitatively). Focus on supporting students’ strategies and having them articulate their ideas (both verbally and in writing), rather than on finding the numerical solution. If the whole class is struggling on the same issue, write relevant questions on the board.

Monitoring Student Progress: There are many ways to gauge understanding and check students’ thinking and reasoning. Below are a variety of questions to consider as you (the teacher) monitor students’ progress on the Transformations Using Triangles and a Circle problem.

- What mathematics do students choose to use? Do they measure the lengths of the sides of the triangles? Do they draw auxiliary lines? Do they use similar triangles? Do they use algebra? Do they use proportion?
- Do students fully explain their solutions?
Revisit criteria students have been developing for critiquing other students’ reasoning (SMP3); add criteria that may emerge through this discussion.

Assessment

- Formative Assessment: Gauging understanding and problem-solving as you visit groups; use Discussion Questions.
- Formative Assessment: Problem-Solving Collaboration and Monitoring Student Progress provides many questions to check for student thinking and gauge understanding.

- Formative Assessment: 3-2-1 Reflection

Lesson Details (including but not limited to:)

Lesson Opening

As we continue to reinforce our understanding of transformations as a vehicle for analyzing and modeling problems involving similarity, we will investigate two new problems today. One involves a popular phenomenon involving multiple iterations of triangles developed by a famous mathematician, Sierpinski, and the other focuses on the relationships found in Transformations Using Triangles and a Circle.

Warm-Up (10 min)
- Revisit the Ticket to Leave questions from Lesson 5. Post (or project) a list of real-world applications that students offered from the collection of responses you collected. Discuss a few; have students share highlights of their ideas.
- Have students pair up, switch their homework with their partners, and ask each other questions to prompt sharing their thinking.

During the Lesson

Sierpinski’s Triangle (20-30 min)
Students, working in pairs, will use ruler and protractor to investigate Sierpinski’s Triangle (use Lesson 6 Handout 1: Justifying Similarity in Sierpinski’s Triangle) (SMP2 - Reason abstractly and quantitatively). When teams complete the tactile experience, have them revisit the investigation using technology (Geometer’s Sketchpad or other geometric software tool (SMP 5 Use appropriate tools strategically). Challenge teams to predict and discover the results for the next set of iterations, and for future sets of iterations. Conclude with teams sharing different findings with entire class.

Guiding Question for Discussion:
- In each iteration of Sierpinski’s Triangle, how are the perimeters and areas of the triangles related to the scale factor?

Extension Questions:
- Would you get similar results if you started with a parallelogram, rather than a triangle? How do you know?
- Would you get similar results if Triangle C was created using points 1/3 of the way along each side of Triangle A? Explain your reasoning.

Introduction to Transformations Using Triangles and a Circle (5 min)

Have students read Transformations Using Triangles and a Circle individually and answer the questions below (Use Lesson 6: Transformations Using Triangles and a Circle handout). This assignment could also have been given as a homework assignment the prior night or as a closure activity in a prior lesson. If students want to attempt a solution at this point it’s fine, but not necessary. Use these questions simply as an introduction.

1. What do you know about the problem?
2. What do you need to find out in order to answer the questions?

Problem-Solving Collaboration (15 minutes)

Students will work on finding a solution in small groups. Encourage students to use the plans they have created individually to work together to find a solution, and then write their solution on chart paper for other groups to see and understand. Students may want to create a “rough draft” of their solutions on paper before transferring them onto chart paper.

Lesson Closing
Today we extended the use of transformations to proving similarity, by making connections to a variety of other mathematical concepts, such as area, perimeter, and relationships in figures. Tomorrow, we will continue our study by solving problems that can be modeled by similar triangles, and an important part of this process will be explaining our reasoning and analyzing each other’s reasoning.

3-2-1 Reflection (5-10 min)
Have each student take a few minutes to write 3 ideas they learned and/or found interesting today, 2 new ideas they learned and/or viewed from a different perspective, and 1 question that is puzzling or that they would like to explore further (on paper or index card). Collect responses as formative assessment; share any highlights as whole class if you desire.

Ticket to Leave (5-10 min)
Allow groups to work on the problem until the last 5-7 minutes, when they should complete the Ticket to Leave (below). Also, give students a chance to review and briefly discuss their understanding of the Homework problem before they leave; confirm ideas/discuss assignment as a class.

Is there more than one solution process to the Transformations Using Triangles and a Circle problem? Why do you think so (or not)?

Extended Learning/Practice (homework)
Extension: The Wonderful World of Fractals
Students explore fractals through a variety of examples (in nature, computer-generated, etc.). Students explore a range of pictures, video, and mathematical problems underlying fractal designs.

Enrichment: Have students conduct further research on the foundation of Sierpinski’s Triangle, evidence of fractals in nature, or other phenomena that exhibit similarity in the real world.
Lesson 6 Handout 1: Justifying Similarity in Sierpinski’s Triangle

Sierpinski’s Triangle is named after the mathematician who noticed interesting patterns in similar triangles. Conduct this investigation with a partner. Record both your results and your reasoning about the observations you make.

1. Draw a triangle with sides greater than 8 inches, and label it Triangle A. Measure the sides and angles of Triangle A, and enter the dimensions in the table.

2. Locate and mark the midpoints of each side of Triangle A.

3. Connect the midpoints.

4. Label the four new triangles B, C, D, and E. Let triangle C be the center triangle.

5. Measure the side lengths and angle measures of each of the four new triangles. Enter the data in the table.

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<thead>
<tr>
<th>Triangle</th>
<th>Angle 1</th>
<th>Angle 2</th>
<th>Angle 3</th>
<th>Side 1</th>
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<th>Side 3</th>
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<td>Triangle A</td>
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6. What is a transformation sequence that
   a. maps Triangle B to Triangle A?
   b. maps Triangle C to Triangle A?

7. What true statements can you make about all four triangles formed during the first iteration (Triangles B, C, D, and E)?

8. What true statements can you make about the perimeters of Triangles B, C, D, and E, as compared to the perimeter of Triangle A?

9. What true statements can you make about the areas of B, C, D, and E, as compared to the area of Triangle A?

10. Based on your observations and data recorded above, make conjectures about successive iterations of this process (in other words, if you were to repeat experiments 2-5 for Triangles B, C, D, and E and extend the data in your chart to the new triangles formed by further iterations). Explain your reasoning and provide evidence for your conclusions.
Lesson 6 Handout 2: Transformations Using Triangles and a Circle

The diagram shows an equilateral triangle with a circle inscribed so that it is tangent to the midpoints of the sides of the triangle.

1. Connect the midpoints of the triangle. Is the smaller triangle a dilation of the larger triangle? How do you know? Explain your reasoning.

2. Determine the transformations needed so that the resulting triangle is a dilation of the original. What would the scale factor be? Justify your conclusion with diagram(s) and explanation.
Sample Answers for Handout 1:

(for the table) The sum of the measures of Angle 1, Angle 2, and Angle 3 must be 180 for all triangles. The side ratio for each side in Triangle A should be in a 2:1 ratio to the corresponding sides of Triangles B, C, D and E.

1. a. a translation and a dilation  
   b. a reflection, a translation and a dilation
2. The triangles are similar to Triangle A. The triangles B, C, D, and E are congruent.
3. Since the sides are in a 2:1 ratio, the perimeter of Triangle A must be in a 2:1 ratio to the perimeters of each of the other triangles.
4. Since the side ratio is 2:1, the area ratio must be 4:1. This makes sense since we have divided Triangle A into 4 congruent triangles.
5. Since we began with a random triangle, this implies that any triangle, when divided this way, will create 4 congruent triangles...

Sample Answers for Handout 2:

1. It is not a dilation.
2. The small triangle can be reflected over any of its sides, or  
   a. rotated 60, 180, or 300 degrees. The scale factor is $\frac{1}{2}$. 
Lesson 7 – Using Similarity to Solve Problems

Time (minutes): 60

Overview of the Lesson
In lessons 7 – 9, students apply their understanding of similarity through geometric transformations to investigate a variety of real-world problems, some of which may be familiar and traditionally solved by Euclidean Geometry. Students apply their newly developed lens of transformational geometry through reasoning about similarity and critiquing each other’s reasoning, as well as using their analysis of given solutions to problem to produce alternative solutions. Problems include finding the height of a tree using its shadow, a “bank shot” pool table problem, and a “breakout” video game problem. As you plan, consider the variability of learners in your class and make adaptations as necessary.

Standard(s)/Unit Goal(s) to be addressed in this lesson:
G-SRT.2 Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
G-SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.
SMP2 Reason abstractly and quantitatively.
SMP3 Construct viable arguments and critique the reasoning of others.
W.2 Produce clear and coherent writing in which the development, organization, and style are appropriate to task, purpose, and audience.

Essential Question(s) addressed in this lesson:
How can similarity be modeled in real life situations?
What is the value of logical reasoning in constructing an argument?

Objectives
- Apply and construct similar triangles to solve problems.
- Draw on and apply understanding of similarity theorems in real-world context, particularly, the AA Theorem.

Language Objectives
Targeted Academic Language

What students should know and be able to do before starting this lesson
Students should be comfortable solving for missing sides of pairs of similar triangles and building necessary equations to model the problems. They should recall and recognize the value of the AA postulate in determining whether pairs of triangles are similar and what similarity means in terms of corresponding angles and sides.

Anticipated Student Pre-conceptions/Misconceptions
Students may have difficulty interpreting the problem, which could produce incorrect drawings of the triangles necessary to solve the problem.

Instructional Materials/Resources/Tools
- Protractor and ruler
- Geometer’s Sketchpad or free online geometric tool (e.g., GeoGebra www.geogebra.com or Draw Island www.drawisland.com)

Instructional Tips/Strategies/Suggestions for Teacher
Warm Up: In this problem, students apply the AA postulate to the figure. Students should also be paying attention to notation involved in aligning corresponding vertices.

Some students may have difficulty seeing the similar triangles. Encourage students first to identify and re-draw the three triangles in the figure and then to mark the congruent angles with congruent angle notation. Some may find this easier if they think of the three sides of the triangles as “short leg”, “long leg” and “hypotenuse”. This helps students recognize the AA postulate in context – without measuring.

The problems in Lesson 7 Handout are generally traditional, but they are included in order to provide direct preparation for the CEPA. The purpose of including them here is not to repeat traditional, formulaic problem-solving methods, but rather, to invite students to see and solve these problems from different perspectives based on their new understanding, similarity through transformations. These problems are also vehicles for further practice with expressing and critiquing each other’s reasoning. (SMP.3- Construct viable arguments and critique the reasoning of others.)
As students work through the problems, the accuracy of their diagram, based on their interpretation of the problem, is the key to developing appropriate solutions. Encourage them to identify the two corresponding congruent angles first, and then re-draw triangles with corresponding parts in the same position.

Students often make errors setting up the proportion of the corresponding sides. Using the location of the side may prove helpful. For example, compare the small triangle’s left side to the large triangle’s left side. In addition, stress the consistency in setting up each fraction, the numerator of each fraction must be a side of the same triangle and the denominator of each triangle from the other triangle.

Students have been collecting criteria and refining their methods for critiquing each other’s reasoning throughout this unit (SMP3-Construct viable arguments and critique the reasoning of others). Have students add their ideas to the Carousel Sharing activity.

Remember that the focus of this unit is on **transformational geometry**. Students should have facility with explaining which transformations need to occur to align similar figures. You may want to use the following prompt.

When students share their final Presentations, encourage them to use the following questions in critiquing each other’s reasoning:

- Did you use the same method for solving? What was different? Did you get the same solution?
- Describe the strategy that the other team used.
- Rate your teams’ and the other team’s solutions and reasoning for accuracy, clarity, sufficient evidence, and/or other criteria you have been collecting during this unit.
- Which was the most efficient method? Explain.
- What transformations can be used to align the similar triangles?

**Assessment**

- Pre-Assessment: Warm-Up
- Formative Assessment: Check for understanding and ability to express reasoning during problem-solving and presentations.
Lesson Details (including but not limited to: )

Lesson Opening
Now that we have investigated and gained a deeper understanding of transformational geometry, the next few lessons will prepare you for the final performance assessment, in which you will apply your learning to design a mini-golf course. Now we will do more work involving constructing similar triangles, continuing to use technology, in order to solve problems.

Warm-Up (10-15 min) (SMP 2: Reason abstractly and quantitatively)
Refer to diagram at the right. Pose the following problem to students, working individually at first:

Refer to diagram at the right. Pose the following problem to students, working individually at first:

Given a right triangle with the altitude drawn to the hypotenuse, determine the three similar triangles based on the AA postulate. Write similarity statements for each using your understanding of transformations.

Pair-Share: Have students share with a partner their Warm-Up conclusions, and then work with their partner to re-draw each triangle so that corresponding parts are in the same position. Next, have students list the rigid transformations used to re-align the triangles. Ask them to recall their ideas from prior learning, when they explored this same figure.

Whole Class: Encourage students to present their similarity statements and the sequence of rigid transformations used to align the triangles noting any similarities/differences.

Warm-Up Answer Key

Possible solution:
\(\triangle ABC \sim \triangle DBA\)
\(\triangle DAC \sim \triangle ABC\)
\(\triangle DBA \sim \triangle DAC\)

Rigid transformations used to re-align the triangles:

- DBA: Reflect over BA, rotate counterclockwise, translate right
• ADC: Reflect across AC, rotate clockwise, translate up

During the Lesson

Problem-Solving (20 min)
Students engage, in groups of 3, in problem-solving (use Lesson 7 Handout: Solving Problems using Similar Triangles). They should use measuring tools and/or technology (see Resources). While students are working and, later, when you conclude with a whole class discussion, guide the discussion and encourage their thinking.

Discussion Prompts: What is the scale factor of the two triangles? Is there a dilation? Why or why not? If so, what is the center of dilation? Is the dilation negative? (SMP 2 - Reason abstractly and quantitatively)

Lesson Closing

We gained a wide range of solutions today. The work you did on examining and critiquing each other’s reasoning was excellent, and it was not easy. It required you to be precise, analytical, and to consider a different perspective than your own. Keep this lens in mind as we approach the culmination of this unit, as we explore a few more robust problems about triangle similarity that have applications in the real world.

Presentations (10-15 min)
Conclude by having teams display their solutions on flipchart/poster paper (or project them) and verbally present their reasoning to the class. Have teams switch each other’s paper and review the work of one other team. Solutions should include the triangles used with the corresponding congruent angles marked, an extended proportion relating corresponding sides, and the equation used to solve the problem. These posters can be shared among different groups so that they may critique each other's reasoning using the Critique Sheet.

Preparation for CEPA
Remind students of the upcoming CEPA and that the next two lessons will further prepare them to apply their learning to completion of the CEPA.
Lesson 7 Handout: Solving Problems using Similar Triangles

Using appropriate tools or technology, solve the following problems. Begin with accurate diagrams (with paper or online) to represent each problem.

Problem 1
You want to find the width of a deep river. Directly across from you, on the parallel river bank, is a big rock. You decide to investigate the situation by walking 4 feet to the right, and you place a small pole into the ground. Then you proceed 6 more feet to the right. At this point, you make a 90 degree turn and walk 8 ft away from the river bank before turning to face the river, again. At this location you can see directly across the river, as well as over the pole, to the rock. Find the width of the river. Justify your conclusions using transformational geometry.

Problem 2
You need to find the height of a tree. You stand parallel to a tree, an unknown distance away. Both you and the tree are perpendicular to level ground. Parallel rays of sun hit the top of your head and the top of the tree, and then the rays extend to the ground. Find the height of the tree, given the information below. Justify your conclusion using transformational geometry.

1. Your height is 5’3”.
2. Your shadow is 3 ft long.
3. The tree’s shadow is 12 ft long.
Lesson 7 Handout Answer Key

Problem 1

Width of the river: \( \frac{4}{6} = \frac{x}{8} \quad x = \frac{16}{3} \text{ ft} \)

Justification using transformational geometry: Similar triangles; reflect, rotate, translate

Scale factor of the two triangles: Possible answers: \( \frac{2}{3}, \frac{3}{2} \)

Is there a dilation? If so, where is the center of dilation: Yes, there is a negative dilation, at the common vertex of the two triangles.

Problem 2

Width of the river: \( \frac{12}{3} = \frac{x}{5.25} \quad x = 21 \text{ ft} \)

Justification using transformational geometry: translation of small triangle onto larger triangle, aligned at right angle.

Scale factor of the two triangles: Possible answers: \( 4, \frac{1}{4} \)

Is there a dilation? If so, where is the center of dilation: Yes, there is a dilation, at the point where lines drawn through the upper vertices and through the right angles (and other vertices) meet.
**Critique Sheet**

Use this sheet to critique another student or team's problem-solving work. Explain your rationale in the comments section as clearly as possible.

Your/Your Team's Name: ______________________________________________________

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Lesson 8 – Bank Shot, Part 1
Time (minutes): 60
Overview of the Lesson
In this lesson, students apply their understanding of similarity through geometric transformations to investigate a “bank shot” pool table problem. As you plan, consider the variability of learners in your class and make adaptations as necessary.

Standard(s)/Unit Goal(s) to be addressed in this lesson:
G-SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.
SMP2 Reason abstractly and quantitatively.
SMP5 Use appropriate tools strategically.
R.8 Delineate and evaluate the argument and specific claims in a text, including the validity of the reasoning as well as the relevance and sufficiency of the evidence.
W.2 Produce clear and coherent writing in which the development, organization, and style are appropriate to task, purpose, and audience.

Essential Question(s) addressed in this lesson:
How can similarity be modeled in real life situations?

Objectives
- Apply similarity to the solution of problems.
- Recognize similarity in unfamiliar contexts.

Language Objectives

Targeted Academic Language
Angle of Incidence, Angle of Reflection
What students should know and be able to do before starting this lesson

Students should have had some experience using auxiliary lines to help them solve problems involving similar triangles. They will build on their understanding of similarity and their growing ability to reason about their observations and conclusions.

Anticipated Student Pre-conceptions/Misconceptions

Students may not readily see that the Bank Shot problem is a similarity problem involving right triangles. *It is important that they come to this notion on their own.*

Instructional Materials/Resources/Tools

- Marbles, or other small balls
- Box lids (a paper box lid is fine)
- Graph paper (optional)
- Calculator (optional)

Instructional Tips/Strategies/Suggestions for Teacher

This is the first part of a culminating lesson, designed to further prepare students more directly for the CEPA, and to demonstrate students’ understanding of similarity and similarity transformations as it applies to real-world problems.

Problem Solving: *Marbles and Lids:* For students who have not yet taken Physics, the terms “angle of incidence” and “angle of reflection” may be new. Students will conjecture that the angles are congruent, but this fact is never proven in this lesson. It is safe for students to make this assumption for the sake of this lesson. More important here, as with the rest of the unit, is developing students’ skills in clearly articulating the reasoning behind their conjectures.

*Bank Shot:* The Turn and Talk strategy used is intended to give students time to understand the problem prior to solving it.

The *Bank Shot* problem is best solved in groups of 3. A suggestion would be homogeneous groups, so that specific scaffolding can be provided as needed. Try not to give any hints about similar triangles to students as they attempt the *Bank Shot* problem, especially in the beginning, as they are acquainting themselves with the problem. The discourse will give students the confidence to tackle challenging tasks in the future.
Some students may struggle with the terms “north wall,” “south wall,” etc., even with the compass. It might be helpful for them to transfer the problem onto a coordinate grid.

Students may not readily see that the Bank Shot problem is a similarity problem involving right triangles. It is important that they come to this notion on their own. (SMP.2- Reason abstractly and quantitatively)

Provide a variety of tools, such as calculators and graph paper, and allow students to choose whether or not to use them. Allow students to struggle. (MP.5 Use appropriate tools strategically)

Math Journal: After students have wrestled with the problem, if there are still groups that have not made the connection to similar triangles, the closure discussion should allow them to make that connection.

**Assessment**
Formative Assessment: While students engage in problem-solving in groups, check for understanding and the ways in which they express their reasoning.

**Lesson Details (including but not limited to:)**

**Lesson Opening**

**Problem-Solving: Marbles and Lids (20 min)** (Refer to image at right)

Students work in pairs, rolling a marble inside a paper box lid.

- What are some observations you notice as you roll the ball into one of the sides?
- Try hitting one wall with the marble at different angles. What do you notice about the path of the ball before and after it hits the wall?
- Draw a picture and write an explanation to support your claim.

Explain to students:

The dotted line runs perpendicular to the wall surface (indicated by the horizontal line). Angle $b$ is called the **Angle of Incidence**, and angle $c$ is called the **Angle of Reflection**.
Discuss: (SMP2- Reason abstractly and quantitatively)

- How do you think the angle of incidence and the angle of reflection compare? Does this support your claim about the marble’s path? How?
- Given this assumption, how do the other two angles shown (angles a and d) compare? How do you know?
- What do you expect to happen if you roll the marble perpendicular to the wall? Why? Use your new vocabulary to explain your reasoning.

During the Lesson

Problem-Solving: Bank Shot (20 min)
Use Lesson 8 Handout: Bank Shot Problem.
Have students, at first individually, read the problem and write a few sentences that describe:
- what the problem is about
- what information you know before doing any work
- what you are trying to figure out

Have students turn and talk to a partner about their ideas. Share highlights from group discussions with the whole class, after students have had a few minutes to discuss the problem.

In groups of 3, students work on solving the Bank Shot problem.

Discussion Prompts:
- Where do you see similar triangles in the diagram?
- Using your knowledge of angles of incidence and reflection, where can you insert lines in the diagram to create similar triangles?
- Is it possible/necessary to extend the table to create similar triangles? How do you know? Explain your reasoning.

Lesson Closing

Math Journal (5 min)
Have students consider and respond to the questions below (individually, in a math journal).
How does transformational geometry help you solve this problem?

**Homework**
Students will continue to work on the Bank Shot problem the next day. In the meantime, encourage students to bring ideas that are resonating with them and any questions that are emerging about this problem to the next day’s lesson.
Lesson 8 Handout: Bank Shot Problem
Adapted from Illustrative Mathematics Project (licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License)

Pablo is practicing bank shots on a standard 4 ft.-by-8 ft. pool table that has a wall on each side, a pocket in each corner, and a pocket at the midpoint of each eight-foot side. Pablo places the cue ball one foot away from the south wall of the table and one foot away from the west wall, as shown in the diagram below. He wants to bank the cue ball off of the east wall and into the pocket at the midpoint of the north wall.

At what point should the cue ball hit the east wall? Explain.

Alyssa solves the problem using Strategy A. Brenda uses Strategy B. See their work below.

Strategy A (Alyssa’s Strategy)

Let \( x \) be the distance from the northeast corner of the table at which Pablo wants the cue ball to hit the east wall. We start by drawing two right triangles as shown in the diagram above: one whose hypotenuse is the segment from the desired point of contact with the east wall to the
north pocket, and one whose hypotenuse is the segment from the cue ball to the point of contact with the east wall. The former right triangle has legs of length 4 ft. and \(x\) ft.; the latter has legs of length 7 ft. (because the cue ball begins one foot away from the west wall) and \((3-x)\) ft. (because it begins one foot away from the south wall). Because the angle at which the cue ball hits the wall is equal to the angle at which it rebounds off of it, and because the two right triangles already have one right angle each, we know the two triangles are similar by AA similarity, and therefore \(7(3-x)=4x\). Multiplying each side by \(x(3-x)\) yields \(7x=4(3-x)\) and we obtain \(x = \frac{12}{11}\) ft. Therefore, Pablo wants the cue ball to contact the east wall \(x = \frac{12}{11}\) ft. away from the northeast corner of the table.

**Strategy B (Brenda’s Strategy)**

This alternative solution uses a different strategy, in which we picture the reflection of the pool table across the east wall. Because the angle of incidence is equal to the angle of reflection when the cue ball collides with the east wall, if we reflect the continuation of the ball's trajectory across the east wall, this reflection will form a straight line with the ball’s initial trajectory. We can use this fact to determine exactly where the cue ball should "cross" the east wall in this diagram. (In fact, if a pool table came with mirrors on each wall, Pablo could use this strategy to aim his shot perfectly, since he could simply direct his shot at the north pocket's reflection in the east wall mirror.)

We draw a right triangle whose hypotenuse is the segment from the cue ball to the reflection of the north pocket. The legs of this right triangle are 7 ft. + 4 ft. = 11 ft. and 3 ft., since the cue ball starts one foot away from the west wall and one foot away from the south wall. Let the distance from the desired point of contact with the east wall and the horizontal leg of this right triangle be \(y\) feet. Then our right triangle contains another, similar right triangle (by AA similarity) whose legs have length \(y\) ft. and 7 ft. Therefore, the scale factor between this smaller triangle and the original right triangle is \(\frac{7}{11}\). To compute \(y\), we multiply 3 by \(\frac{7}{11}\) to get \(\frac{21}{11}\).
Thus the ball should hit the east wall \( \frac{21}{11} \text{ ft.} \) north of the horizontal dotted line in the diagram. This is \( 3 \cdot \frac{21}{11} = \frac{12}{11} \) feet away from the northeast corner pocket.

**Your turn: Compare and Analyze Strategies A (Alyssa’s work) and B (Brenda’s work)**

1. How do the strategies differ?

2. What are the strengths/weaknesses of each person’s approach?

3. After Pablo practices banking the cue ball off of the east wall, he tries placing the eight-ball two feet from the east wall, as shown in the diagram below, so that if he shoots the cue ball from the same spot as he did before, the cue ball will strike the eight-ball directly and sink the eight-ball into the north pocket. How far from the north wall should Pablo place the eight ball? Explain your reasoning using transformational geometry. [Hint: try to think about the problem the way Brenda did.]

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Sample Answer to problem 3.

\[
\frac{7}{21} = \frac{2}{x}
\]

\[7x = \frac{42}{11}\]

\[x = \frac{42}{77}\]

\[x = \frac{6}{11}\]
Lesson 9 – Bank Shot, Part 2
Time (minutes): 60

Overview of the Lesson
In this lesson, students apply their understanding of similarity through geometric transformations to investigate a “breakout” video game problem. As you plan, consider the variability of learners in your class and make adaptations as necessary.

Standard(s)/Unit Goal(s) to be addressed in this lesson:
G-SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.
SMP2 Reason abstractly and quantitatively.
SMP5 Use appropriate tools strategically.
W2. Produce clear and coherent writing in which the development, organization, and style are appropriate to task, purpose, and audience.
R.8 Delineate and evaluate the argument and specific claims in a text, including the validity of the reasoning as well as the relevance and sufficiency of the evidence.

Essential Question(s) addressed in this lesson:
How can similarity be modeled in real life situations?
What is the value of logical reasoning in constructing an argument?

Objectives
• Apply similarity to the solution of problems.
• Recognize similarity in unfamiliar contexts.

Language Objectives

Targeted Academic Language
What students should know and be able to do before starting this lesson

Students should have had some experience using auxiliary lines to help them solve problems involving similar triangles. They will build on their understanding of similarity and their growing ability to reason about their observations and conclusions.

Anticipated Student Pre-conceptions/Misconceptions

Students learned in the previous lesson to use similar triangles to solve this problem. Students will need to share their findings with others in the class verbally, and in writing. Students who typically struggle with communicating their ideas will need support.

Instructional Materials/Resources/Tools

- Graph paper (optional)
- Calculator (optional)
- Flipchart/poster paper, markers

Instructional Tips/Strategies/Suggestions for Teacher

This is the second part of the culminating lesson that demonstrates students’ understanding of similarity and similarity transformations as it applies to a real world problem.

Warm up: Students should connect the Breakout problem to the Bank Shot (pool table) problem from the previous lesson, by making the realization that similar triangles can be used to solve it. While similar to the pool table problem, Breakout is not as complex, and only requires students to figure out that the two triangles are not similar, in order to answer the question.

Students may need more time to complete the Bank Shot problem.

Developing Exemplars provides tremendous opportunities for students to strengthen their capacity for critiquing each other’s reasoning, as well as their own (SMP2 - Reason abstractly and quantitatively; SMP3 - Construct viable arguments and critique the reasoning of others). Very rich discussion can emerge. Scaffold the writing and presentation for students who need it, with sentence stems, prompts, and visual aids...

Closure: Ideally students should think of similarity as more than just corresponding parts and ratios; including a relationship between dilation and translation, reflection and rotation as well. Are students articulating their thinking in terms of similarity transformations?
**Assessment**
- Pre-Assessment: Warm-Up serves as an informal assessment of students’ understanding of using similar triangles to solve problems.
- Formative Assessment: Assess students’ application of theorems and properties related to similarity as they are solving problems in groups. Get a sense of what new perspectives or a-ha moments they are experiencing, and how they are assimilating their learning from this unit to these problems.

**Lesson Details (including but not limited to: )**

**Lesson Opening**

Warm up: Breakout (10 min)
Have students work on the Breakout video game problem (use Lesson 9 Handout: Breakout), either individually or in pairs.

Discuss:
- How does this problem compare to the Bank Shot problem?
- Are the two triangles similar? Explain using transformational geometry.

**During the Lesson**

Problem-Solving: Bank Shot Problem, Continued (20 min)
Teams will continue their work on the Bank Shot Problem from the previous lesson (refer to Lesson 8 Handout: Bank Shot Problem). Refer to discussion questions from Lesson 9 to continue to support and push students’ thinking and reasoning.

As students are working, look for unique solutions to problem 3 and give groups a heads up if they are selected to present (not all groups will present, but all groups should do a write up on chart paper for others to see). Allow a few different presentations. One presentation might be a solution that many groups have; another might be an alternate solution method that only some students have. If possible, try to showcase a solution that is very unique (ask “Does anyone have a solution unique to these?”).

Developing Exemplars (10-20 min)
Students review each others’ solutions through a Gallery Walk, after the whole class discussion (above), by writing comments on each others’ flipcharts regarding criteria for critique. Based on students’ observations and analysis of their own team’s reasoning and that of
other teams, each team creates its own set of criteria for an “exemplar” solution. They develop, write, and display their exemplar. Select a few groups to present their thinking.

Have students use these questions in developing their exemplars and in critiquing each other’s reasoning:

- Did you use the same method for solving? What was different? Did you get the same solution?
- Describe the strategy that the other team used.
- Rate your teams’ and the other team’s solutions and reasoning for accuracy, clarity, sufficient evidence, and/or other criteria you have been collecting during this unit.
- Which was the most efficient method? Explain.
- What transformations can be used to align the similar triangles?

**Lesson Closing**

**Closure (5-10 min)**

Compare, as a class, the commonalities and differences in solution methods, answers, and reasoning in the exemplars posted by each team.

Ask: How does your interpretation of a problem influence your solution?

**Homework Suggestion**

Further practice with solving problems using similar triangles

Analysis of a collection of problems (without solving) to determine whether similar triangles form the basis for solving
Lesson 9 Handout: Breakout Problem

Naomi is playing a *Breakout* video game on a 7 cm x 10 cm screen (the picture below is not to scale). The ball hits the left wall 2 cm from the bottom of the screen, and bounces toward her slider 3 cm from the left side of the screen. The target is located 1.5 cm from the top of the screen and 2.5 cm from the right of the screen. Will the ball hit the target? How do you know? Show your solution and explain your reasoning.
Sample Answer: Create similar triangles using right angles and angles of reflection.

To find \( \frac{14}{3} \), set up the proportion \( \frac{3}{2} = \frac{7}{x} \). Since the height of the screen is 7 cm, we can get a length of \( \frac{7}{3} \) which will correspond to \( \frac{14}{3} \) from the first triangle, and set up the following proportion:

\[
\frac{14}{7} = \frac{3}{y'} \text{ and we get } y = 3.5.
\]
Since we know that the target is 2.5 cm from the right of the screen, we can draw a line and create another similar triangle with a side length of 1 (3.5-2.5).
We can find $z$, the distance from the top of the screen by using similar triangles again. Since they are all similar we can use the proportion $\frac{3}{2} = \frac{1}{z} \rightarrow z = \frac{2}{3}$. Since the ball passes through a point $\frac{2}{3}$ cm from the top of a screen and not 1.5 cm, we can say that the ball will not hit the target.
CEPA

Curriculum Embedded Performance Assessment

Miniature Golf

Standard(s)/Unit Goal(s) to be addressed in this lesson:

G-SRT.2  Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

G-SRT.5  Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

MP2  Reason abstractly and quantitatively.

MP3  Construct a viable argument and critique the reasoning of others.

MP5  Use appropriate tools strategically.

W.2  Produce clear and coherent writing in which the development, organization, and style are appropriate to task, purpose, and audience.
CEPA: Curriculum-Embedded Performance Assessment
Miniature Golf: Information for Students

Task Overview
A town’s revitalization committee is looking for entertainment options for the community. A survey of the town showed an interest in miniature golf. You and your team of golf course architects are designing a proposal for a golf course which you hope the town planners will accept.

Your task is to design one hole of a mini-golf course. You are given the size and shape of the area, and the position of the starting tee. Your team is required to use at least two obstacles and at least 2 possible paths that result in a hole in one. Your proposal must include a clear explanation of how you used similar triangles to determine where to place the hole on your green, and how a hole in one is possible.

Each team will assemble a different hole, and the collection of all class projects will constitute an entire mini-golf course. You may work in teams of no more than 3.

The Product
Your architectural team must provide to the town’s revitalization committee:

1. A 2D representation of the hole you designed and a possible path of the ball
   a. A scale drawing with obstacles included; drawn precisely with appropriate tools (ruler, etc.)
   b. A key for the scale drawing
   c. Select two similar triangles in your scale drawing and describe how you know they are similar, using transformations. Be specific, and use academic vocabulary to explain your reasoning.

2. A 3D model of the green
   a. The dimensions of the 3D model should be at least 8.5 x 11 inches.
   b. The position of starting tee must be no farther away than 2 inches (on the scale model) from the edge of the green.
   c. Locations of 2 obstacles.
   d. Two (2) possible paths that result in a hole-in-one.

3. A written proposal of your golf course hole, to be evaluated by the other architectural teams.
   Include:
   a. A written justification for the location of the hole on the green, including how and which similar triangles were used in developing the design.
   b. Your team’s mathematical justification for the location of the hole on the green, and how it provides two possible paths that result in a hole in one.

(continued next page)
Evaluation Criteria
Your proposal will be peer reviewed by your colleagues – one other golf course architectural teams. Your architectural team will also evaluate another team. Evaluators will provide a written critique to the team they are evaluating.

Criteria by which teams will judge each other include:

- Clear and accurate explanation and demonstration of the ways that similar triangles are used in order to obtain a hole-in-one pathway
- Verification of similar triangles using similarity transformations
- Apparent logical reasoning
- Appropriate math vocabulary and notation
- Verbal presentation
**Geometry: Similarity Transformations – CEPA Miniature Golf – RUBRIC**

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<td>• There is evidence of using prior knowledge and applying it to the problem.</td>
<td>• There is evidence of drawing on some previous knowledge showing some involvement with the task.</td>
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<td><strong>Reasoning and Proof</strong></td>
<td>• Justification of similarity of triangles is based on similarity transformations. Verification of similar triangles shows a thorough conceptual understanding through clear, detailed and logical progression.</td>
<td>• Justification of similarity of triangles is based on similarity transformations. Verification of similar triangles shows a conceptual understanding through clear and logical progression.</td>
<td>• Justification does not show properties of similarity transformations. Verification of similar triangles shows partial conceptual understanding with some logical reasoning.</td>
<td>• Justification does not show properties of similarity transformations. Verification of similar triangles shows little or no conceptual understanding, with little logical reasoning.</td>
</tr>
<tr>
<td><strong>Communication: Written proposal</strong></td>
<td>• Proposal demonstrates a sense of audience with purpose. Proposal is strongly supported and enhanced by similarity criteria for triangles. Precise math language and symbols are used to communicate ideas succinctly. Proposal shows evidence of abstract or symbolic mathematical representations that analyze relationships, extend thinking and clarify.</td>
<td>• Proposal demonstrates a sense of audience and purpose. Proposal is supported by similarity criteria for triangles. Formal math language is used to share and clarify ideas. Proposal shows appropriate mathematical relationships.</td>
<td>• Proposal demonstrates some awareness of audience and some paraphrasing of the task. Proposal is somewhat supported by similarity criteria for triangles. Some formal math language is used and examples are included. Proposal shows some understanding of mathematical relationships.</td>
<td>• Proposal demonstrates no awareness of audience or purpose. Proposal is not supported by similarity criteria for triangles. Everyday language is used to communicate ideas. Proposal shows minimal understanding of mathematical relationships.</td>
</tr>
<tr>
<td><strong>Representation: 2D and 3D models</strong></td>
<td>• Models show abstract or symbolic mathematical representations (scale drawings and calculations), and are constructed and refined using similarity criteria for triangles. Golf course design exceeds listed specifications. There is a strong correlation between the elements of the 2D and 3D models.</td>
<td>• Models show appropriate and accurate mathematical representations, and are constructed and refined using similarity criteria for triangles. Golf course design models all listed specifications. There is a strong correlation between the elements of the 2D and 3D models.</td>
<td>• Models show some attempt to construct mathematical representations to record and communicate problem solving. Golf course design models some listed specifications. There is some correlation between the elements of the 2D and 3D models.</td>
<td>• Models show no attempt to construct mathematical representations. Attention to golf course design specifications is minimal. There is a weak correlation between the elements of the 2D and 3D models.</td>
</tr>
</tbody>
</table>